Caltech

Machine Learning & Data Mining

CS/CNS/EE 155

Lecture 5:

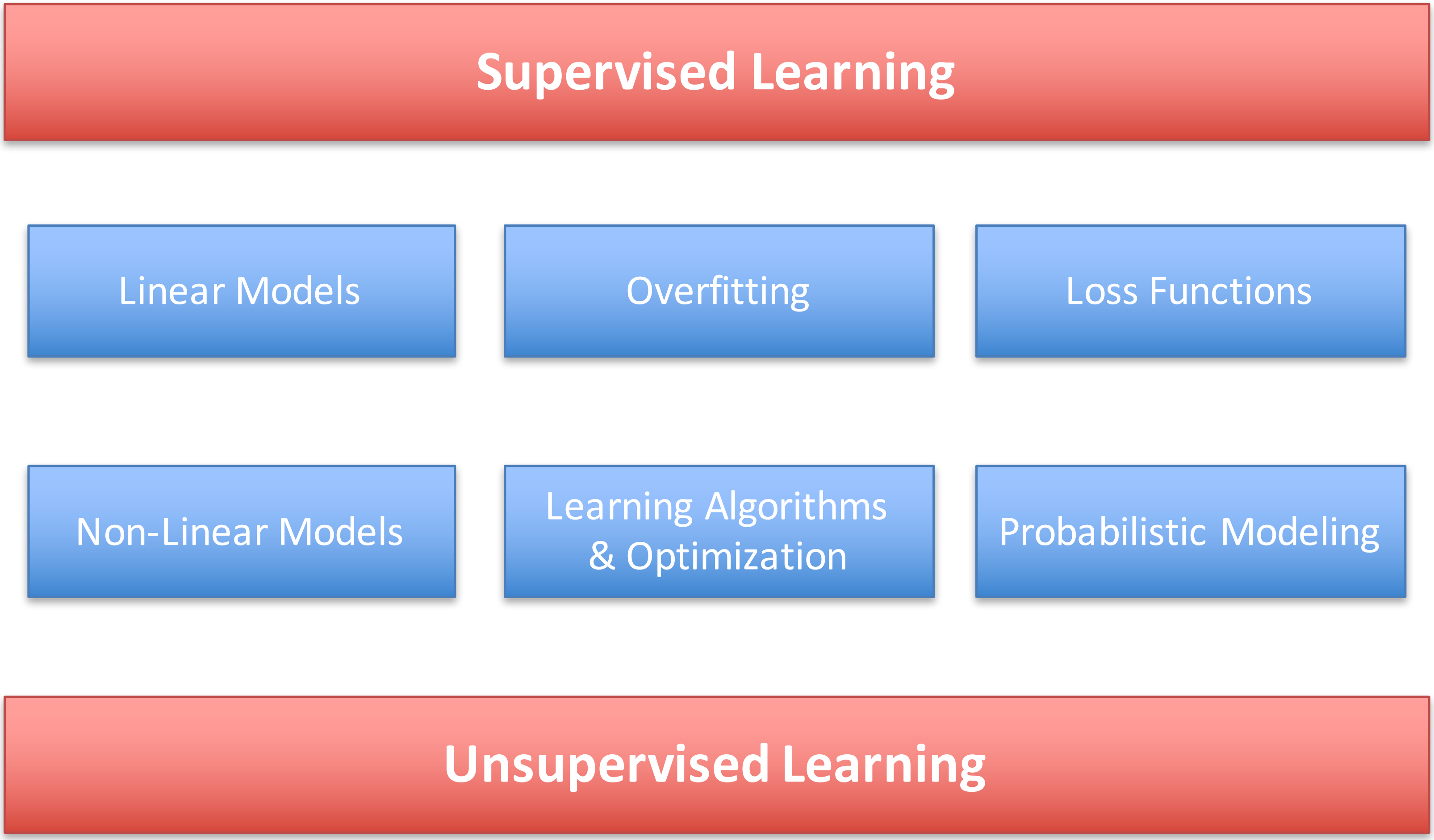
Decision Trees, Bagging &

Random Forests

Announcements

- Homework 2 due tomorrow
  - Some issues arose with Gradescope for HW1
  - We will be posting on Piazza with a list of TODO's
- Homework 3 will be easier than HW1 & HW2
- Kaggle Competition is after Homework 4

Topic Overview



Supervised Learning

Linear Models

Overfitting

Loss Functions

Non-Linear Models

Learning Algorithms & Optimization

Probabilistic Modeling

Unsupervised Learning

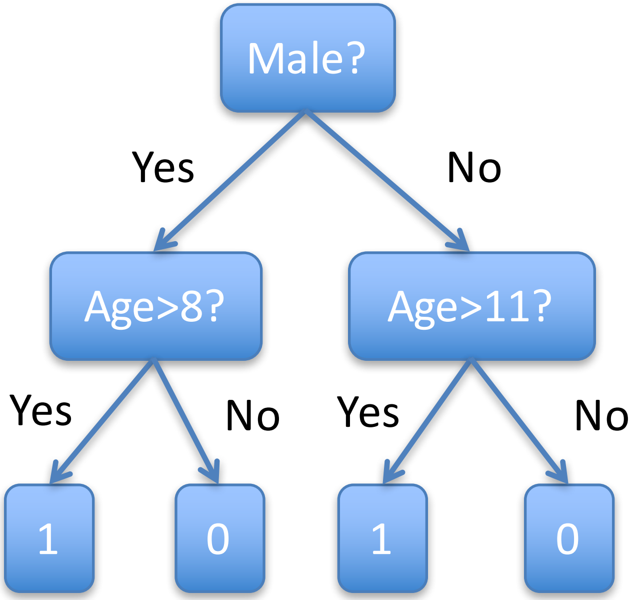
This Lecture

- Focus on achieving highest possible accuracy
  - Decision Trees
  - Bagging
  - Random Forests
  - Highly non-linear models

- Next Lecture
  - Boosting
  - Ensemble Selection

Decision Trees

(Binary) Decision Tree



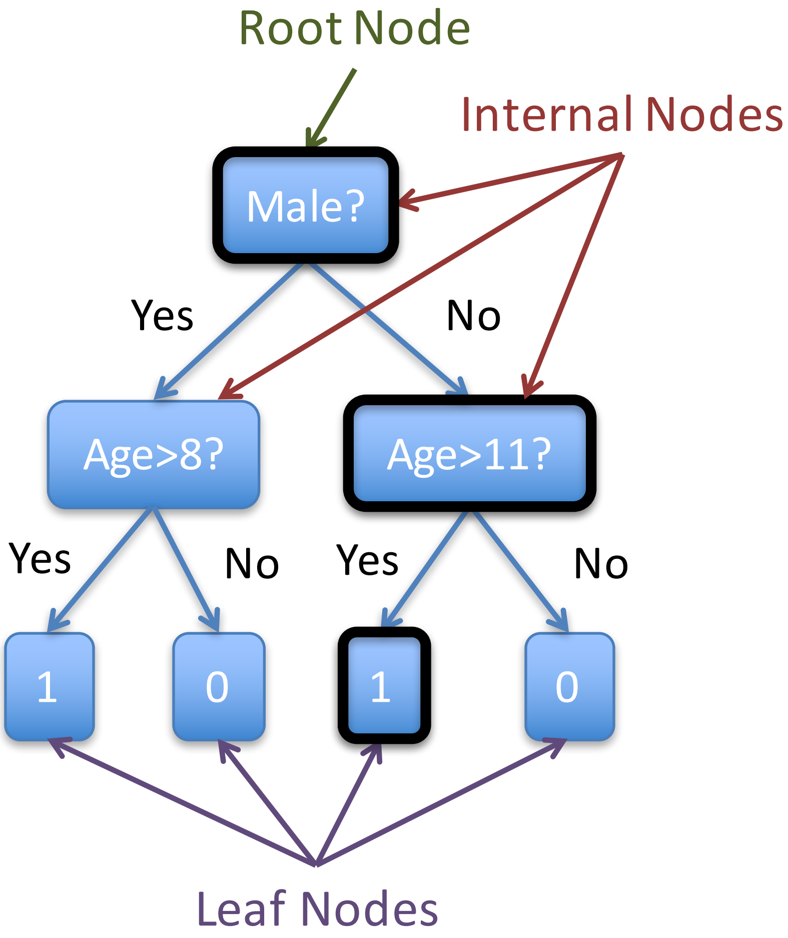
Don’t overthink this, it is literally what it looks like.

Person	Age	Male?	Height > 55”
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	10	0	0

x

y

(Binary) Decision Tree



Input: **Alice**
Gender: Female
Age: 14

Prediction: Height > 55”

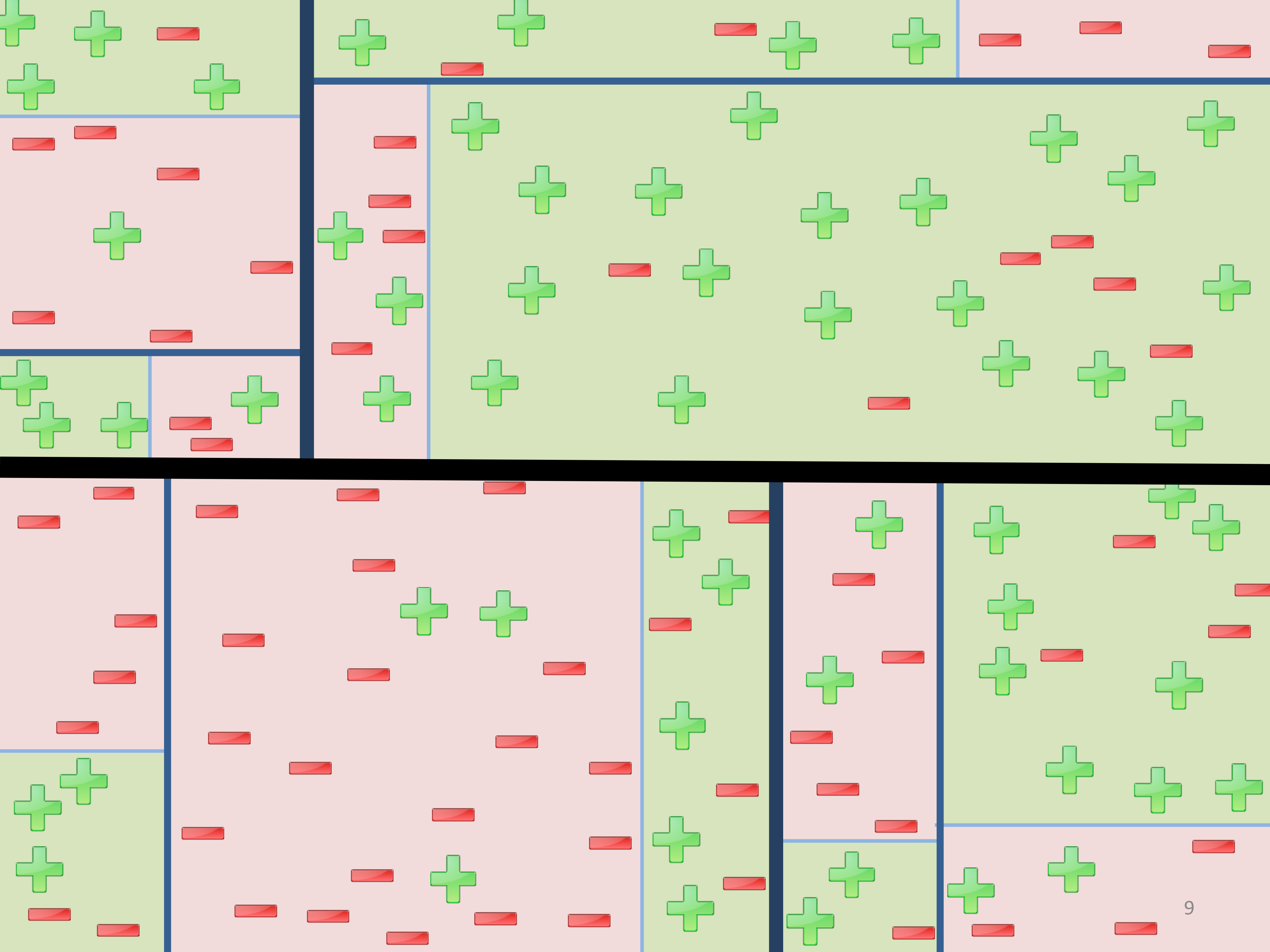
Every **internal node** has a **binary** query function q(x).

Every **leaf node** has a prediction, e.g., 0 or 1.

Prediction starts at **root node**.
Recursively calls query function.
Positive response → Left Child.
Negative response → Right Child.
Repeat until Leaf Node.

Queries

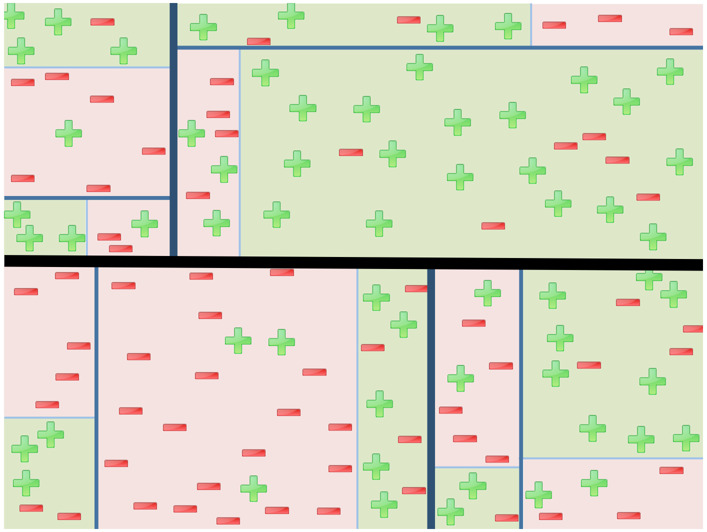
- Decision Tree defined by Tree of Queries
- Binary query q(x) maps features to 0 or 1
- Basic form: $q(x) = \mathbf{1}[x^d > c]$
  - $\mathbf{1}[x^3 > 5]$
  - $\mathbf{1}[x^1 > 0]$
  - $\mathbf{1}[x^{55} > 1.2]$
- Axis aligned partitioning of input space



Basic Decision Tree Function Class

- "Piece-wise Static" Function Class
  - All possible partitionings over feature space.
  - Each partition has a static prediction.

- Partitions axis-aligned
  - E.g., No Diagonals

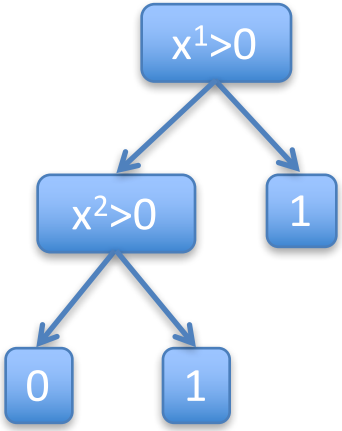


- (Extensions next week)

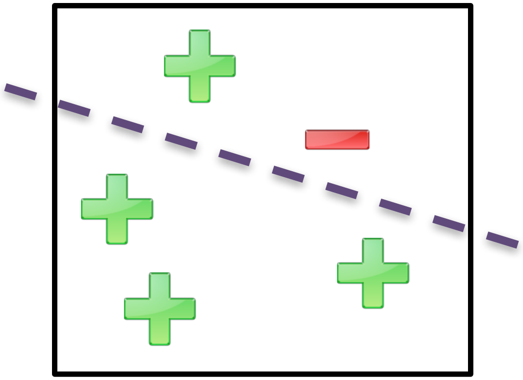
Decision Trees vs Linear Models

- Decision Trees are NON-LINEAR Models!

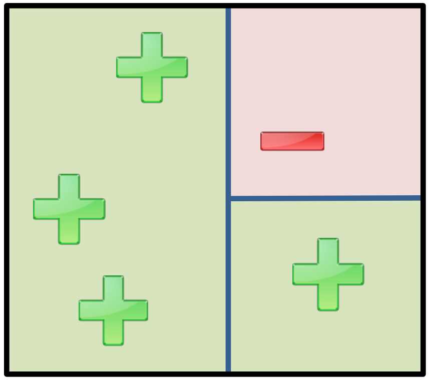
- Example:



No Linear Model
Can Achieve 0 Error



Simple Decision Tree
Can Achieve 0 Error



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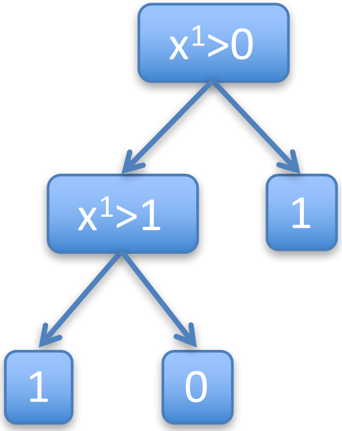

No Linear Model
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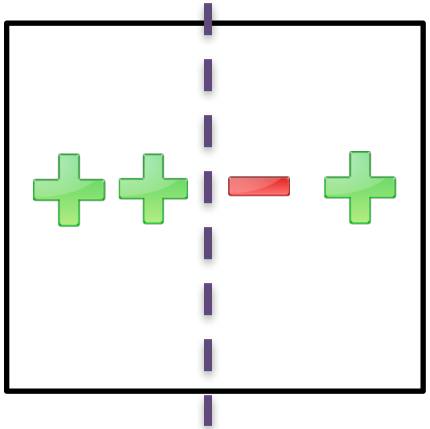
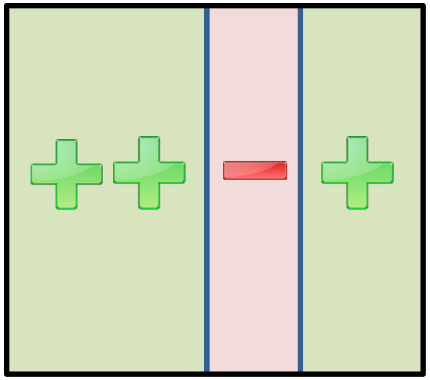


Simple Decision Tree
Can Achieve 0 Error

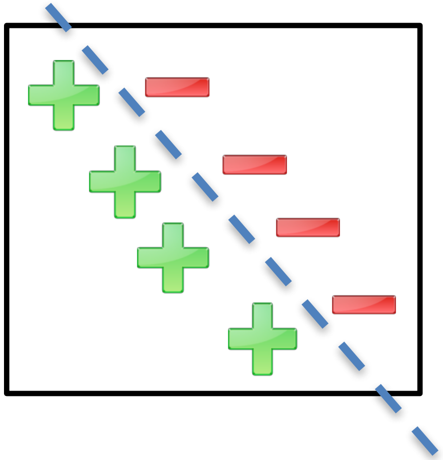


Decision Trees vs Linear Models

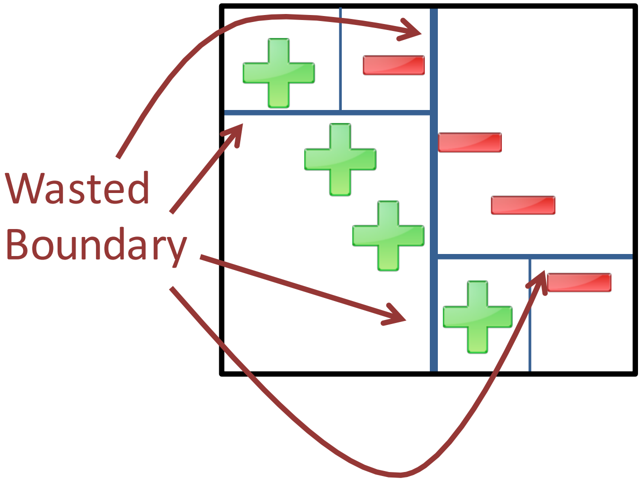
- Decision Trees are AXIS-ALIGNED!
  - – Cannot easily model diagonal boundaries

- Example:

Simple Linear SVM can
Easily Find Max Margin

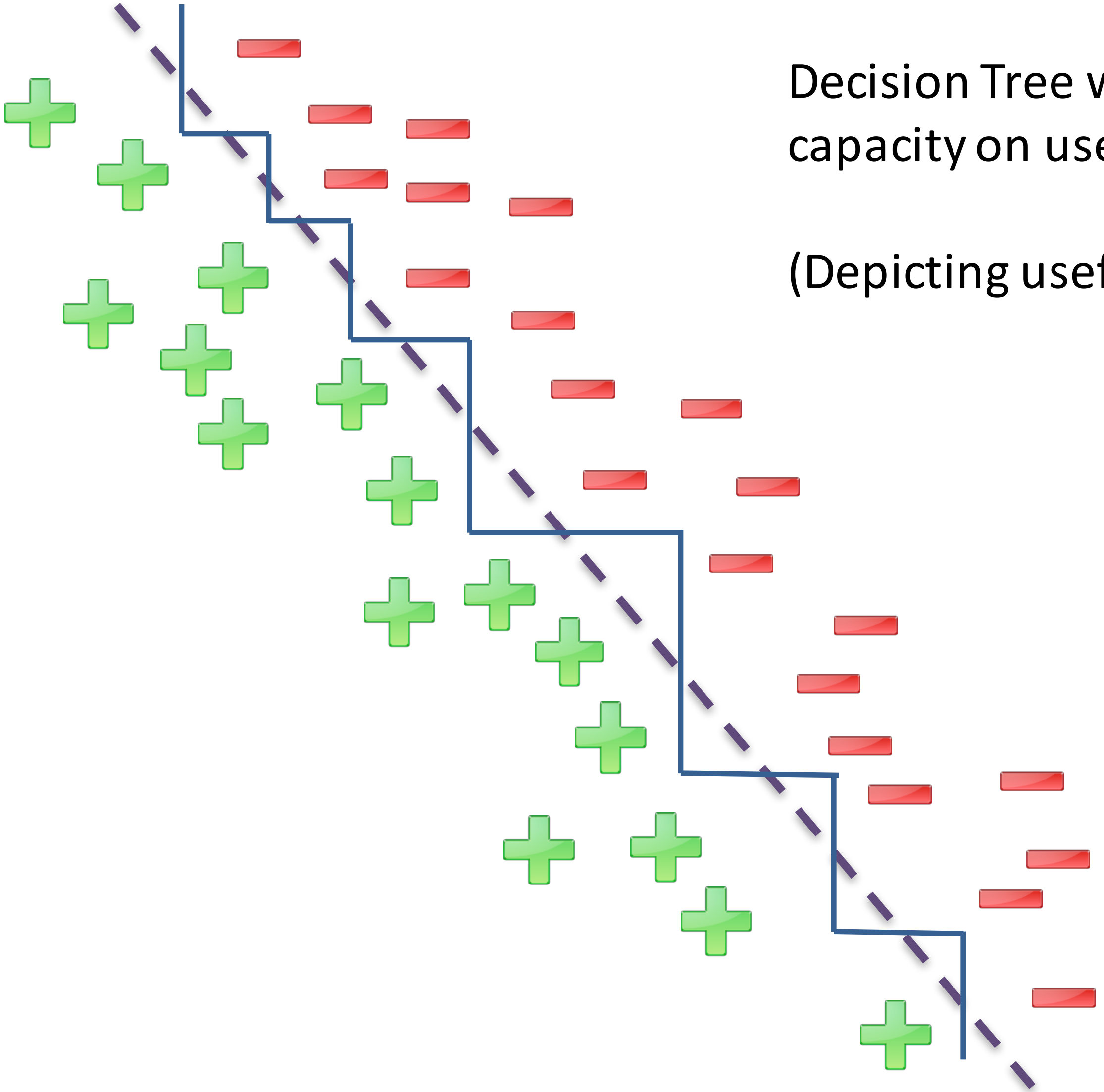


Decision Trees Require
Complex Axis-Aligned
Partitioning



Wasted
Boundary

More Extreme Example



Decision Tree wastes most of model capacity on useless boundaries.

(Depicting useful boundaries)

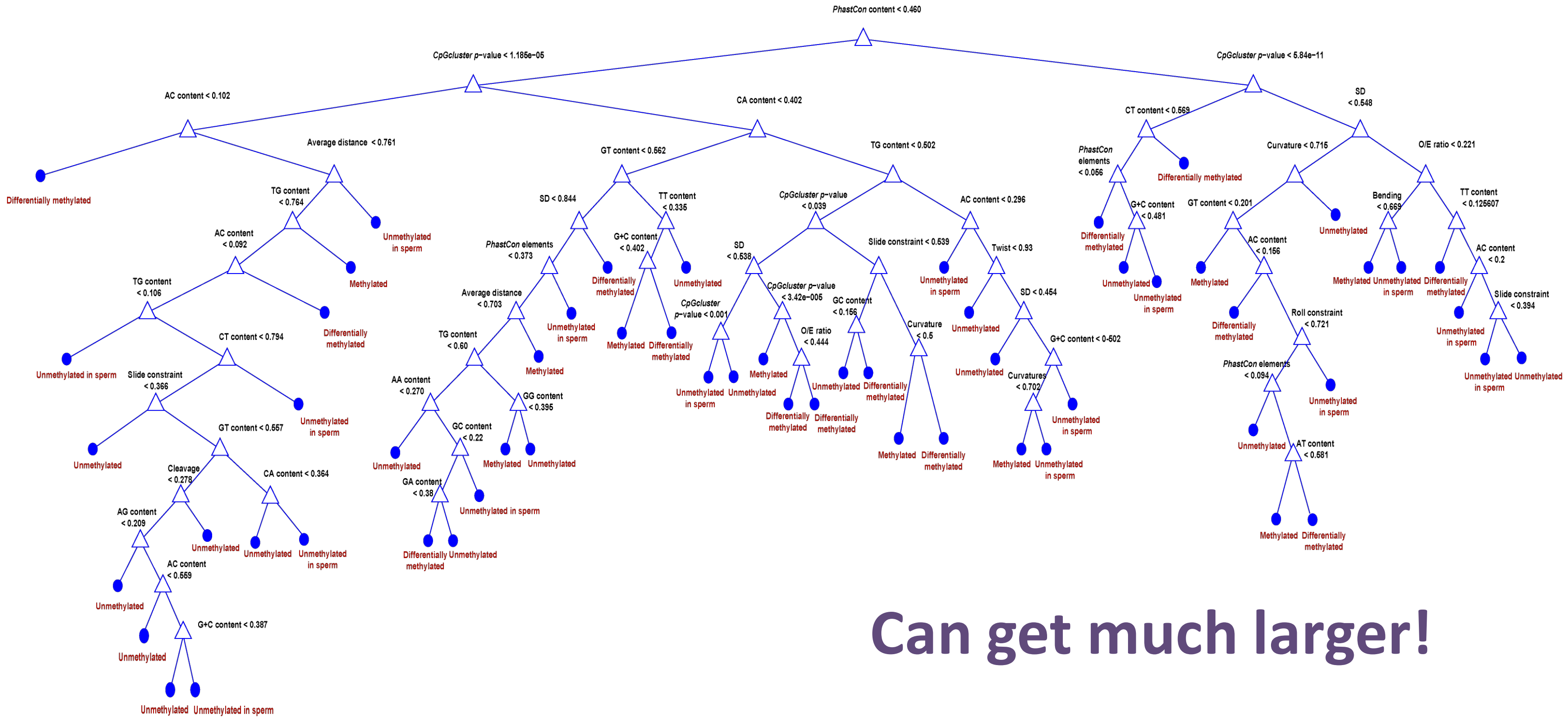
Decision Trees vs Linear Models

- Decision Trees are often more accurate!

- Non-linearity is often more important
  - Just use many axis-aligned boundaries to approximate diagonal boundaries
  - (It's OK to waste model capacity.)

- **Catch:** requires sufficient training data
  - Will become clear later in lecture

Real Decision Trees

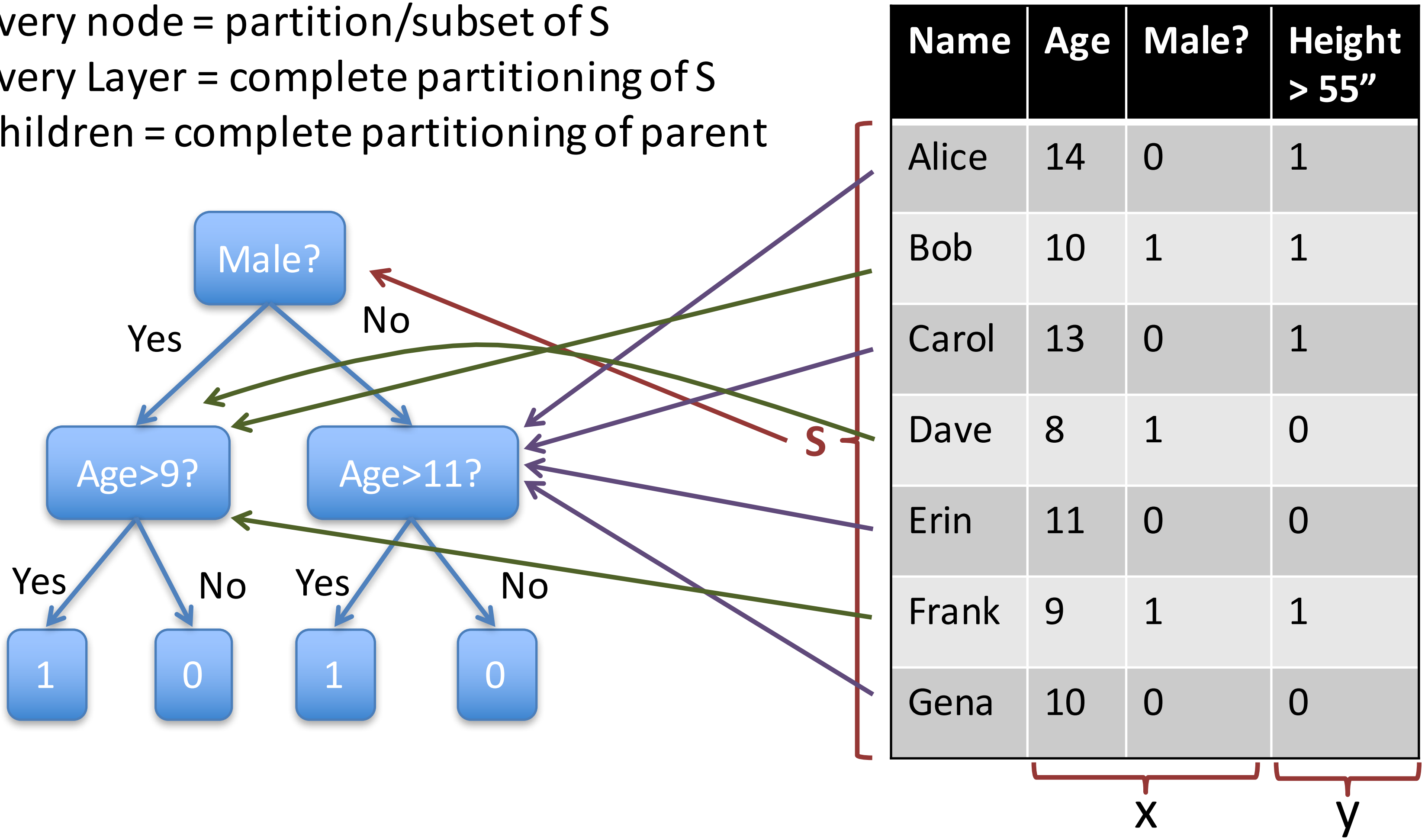


Can get much larger!

Image Source: http://www.biomedcentral.com/1471-2105/10/116

Decision Tree Training

Every node = partition/subset of S
Every Layer = complete partitioning of S
Children = complete partitioning of parent



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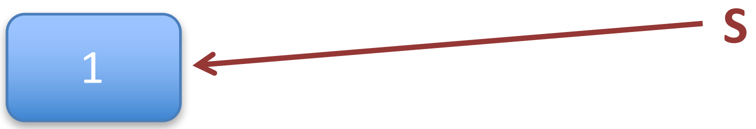
Thought Experiment

- What if just one node?
  - (I.e., just root node)
  - No queries
  - Single prediction for all data



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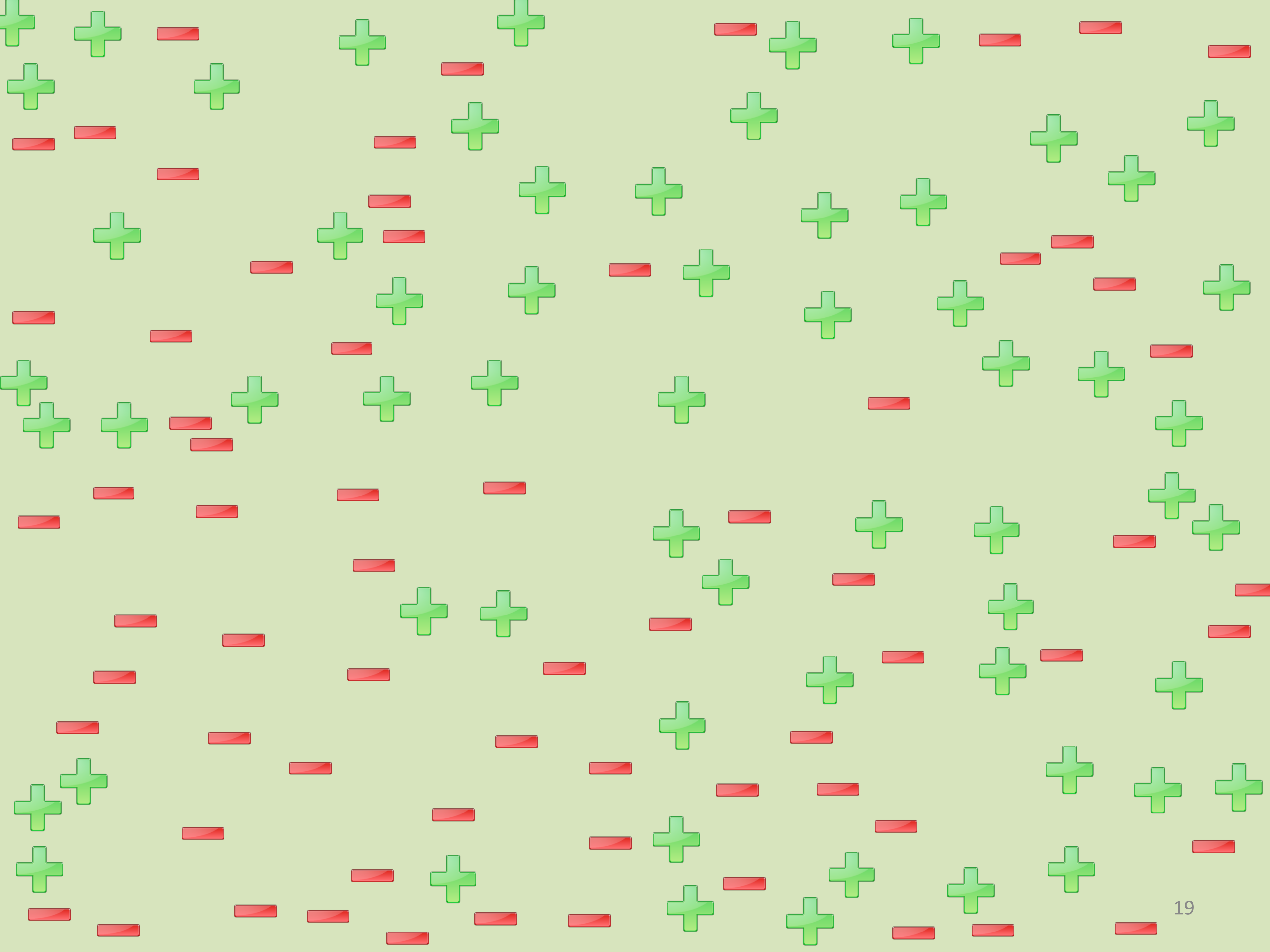
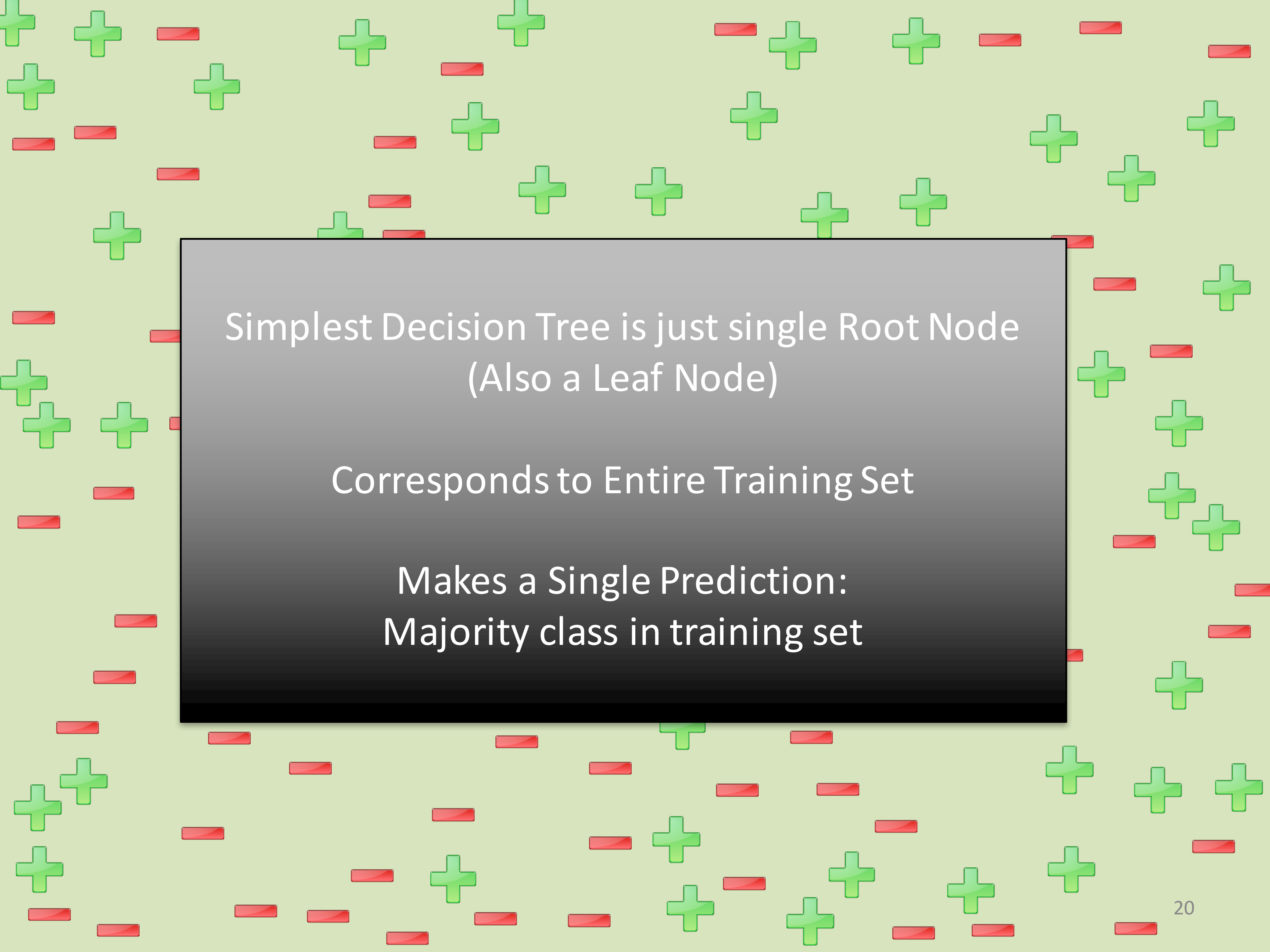
x y





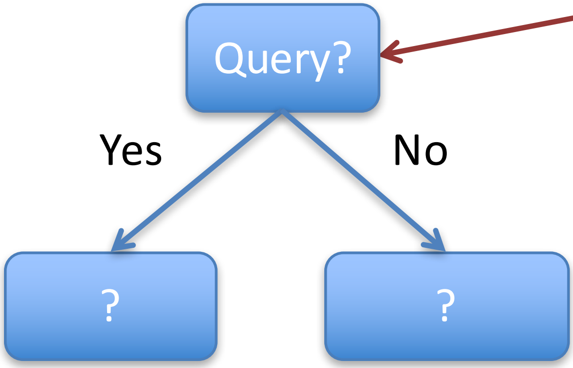
Simplest Decision Tree is just single Root Node (Also a Leaf Node)

Corresponds to Entire Training Set

Makes a Single Prediction:
Majority class in training set

Thought Experiment Continued

- What if 2 Levels?
  - (I.e., root node + 2 children)
  - Single query (which one?)
  - 2 predictions
  - **How many possible queries?**



S

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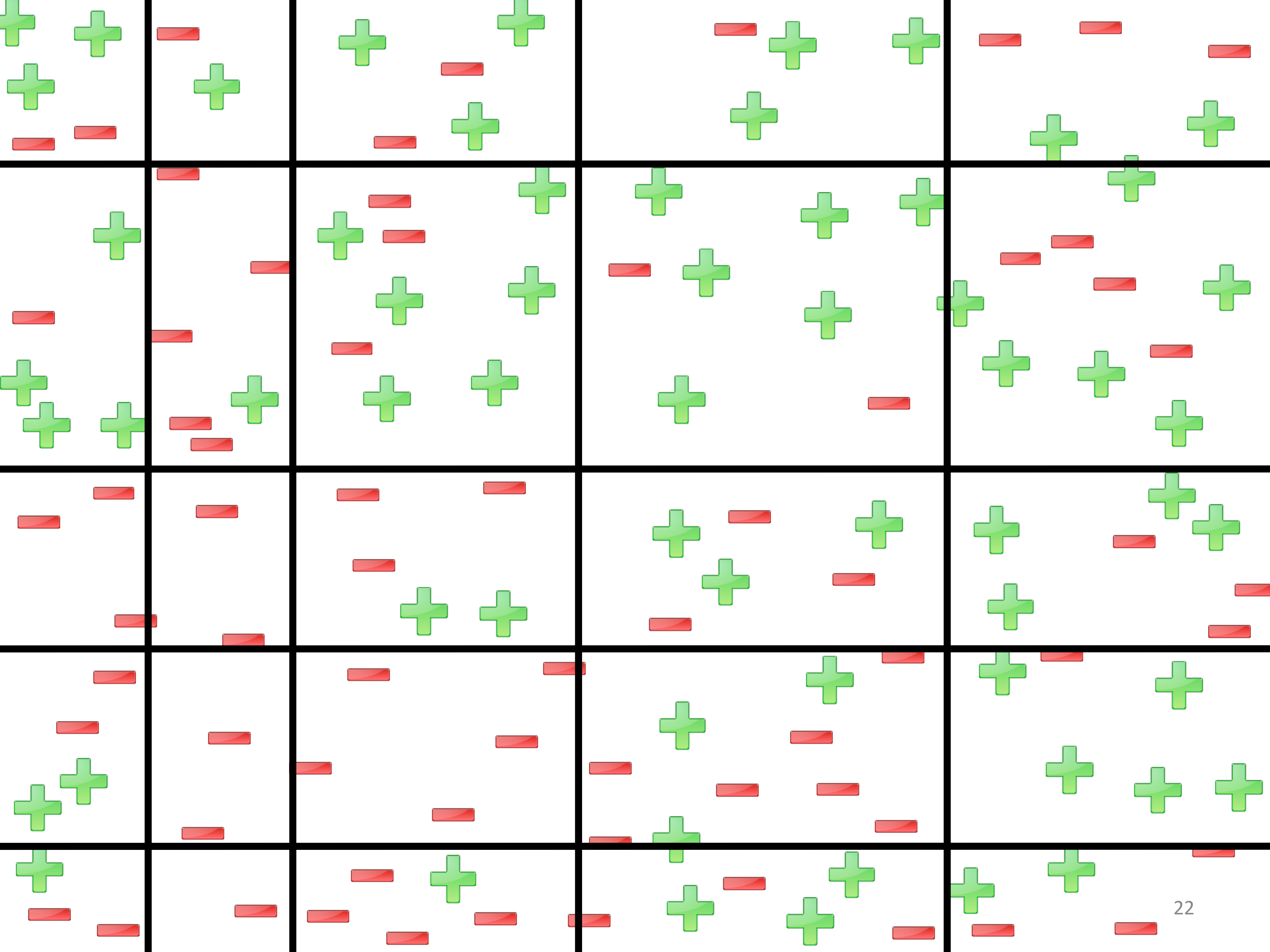
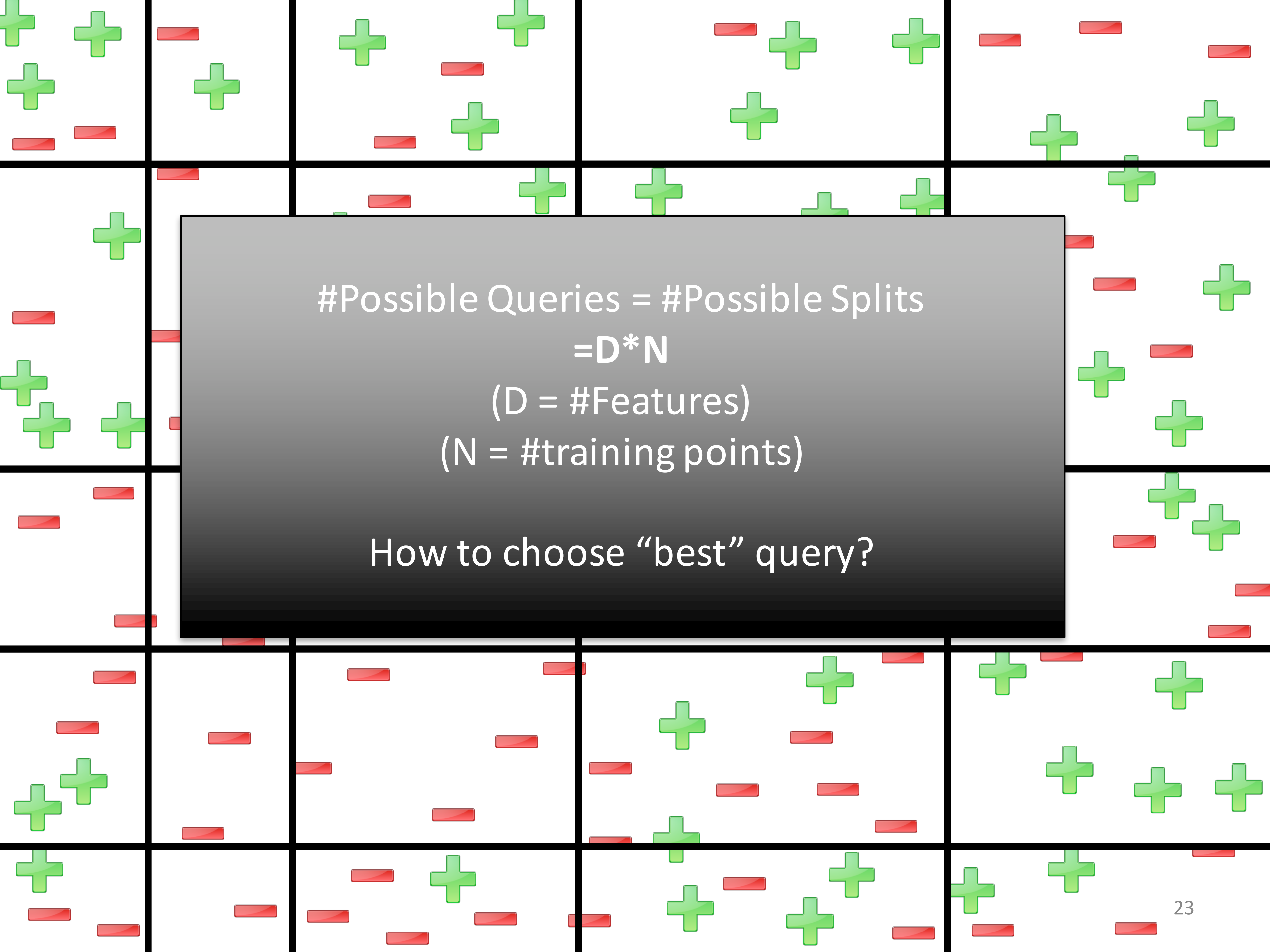
x y





#Possible Queries = #Possible Splits

=D*N

(D = #Features)

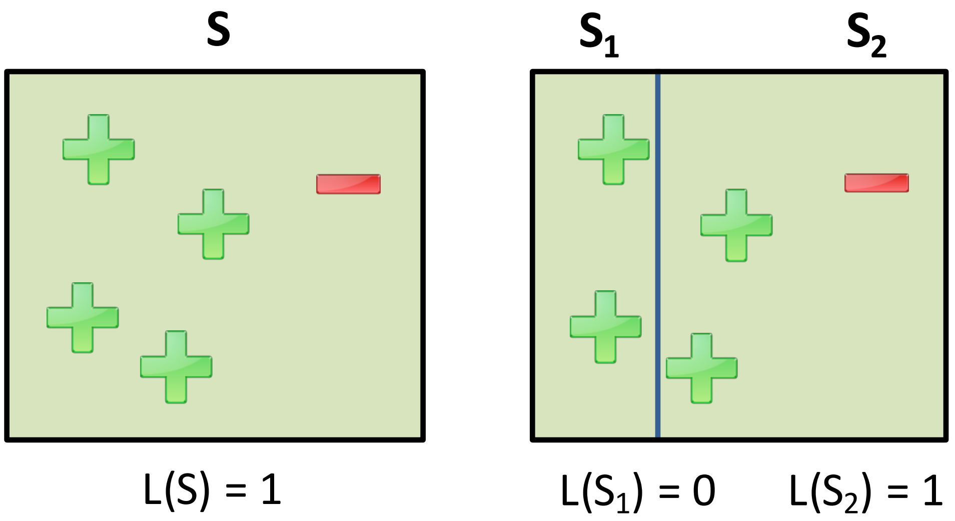
(N = #training points)

How to choose "best" query?

Impurity

- Define impurity function:
  - E.g., 0/1 Loss: $L(S') = \min_{\hat{y}\in\{0,1\}} \sum_{(x,y)\in S'} 1_{[\hat{y}\neq y]}$

Classification Error of best single prediction



Impurity Reduction = 0

No Benefit From This Split!

Impurity

Classification Error of best single prediction

- Define impurity function:
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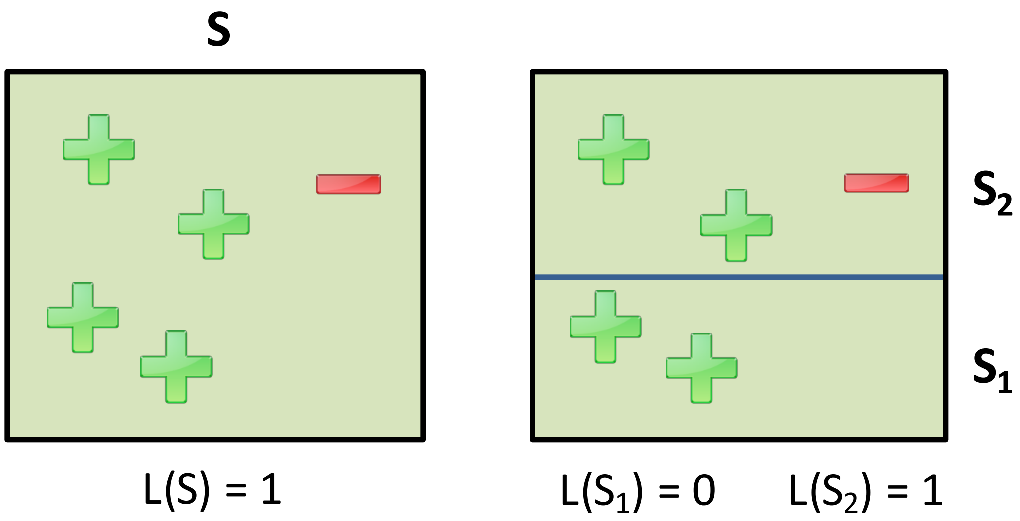


L(S) = 1

$L(S_1) = 0$ $L(S_2) = 1$

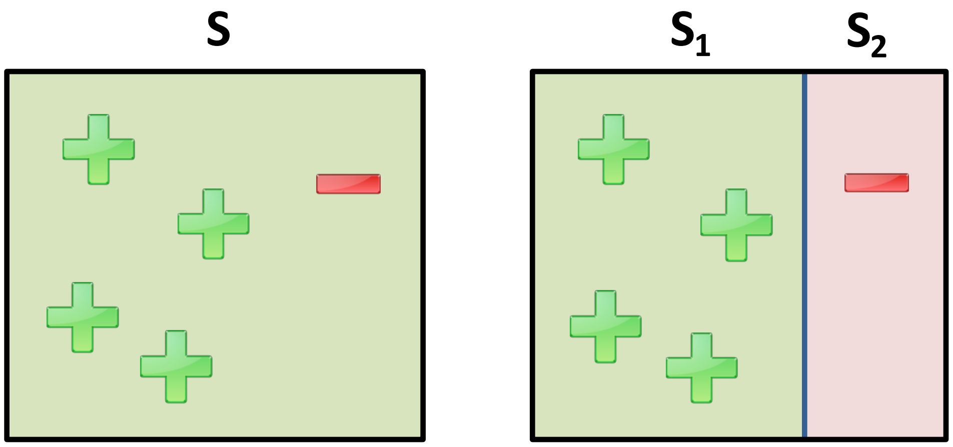
Impurity Reduction = 0

No Benefit From This Split!

Impurity

Classification Error of best single prediction

- Define impurity function:
  - E.g., 0/1 Loss: $L(S') = \min_{\hat{y} \in \{0,1\}} \sum_{(x,y) \in S'} 1_{[\hat{y} \neq y]}$



$L(S) = 1$

$L(S_1) = 0$ $L(S_2) = 0$

Impurity Reduction = 1

Choose Split with largest impurity reduction!

Impurity = Loss Function

- **Training Goal:**
  - Find decision tree with low impurity.

- **Impurity Over Leaf Nodes = Training Loss**

$$L(S) = \sum_{S'} L(S')$$

S' iterates over leaf nodes
Union of S' = S
(Leaf Nodes = partitioning of S)

$$L(S') = \min_{\hat{y} \in \{0,1\}} \sum_{(x,y) \in S'} 1_{[\hat{y} \neq y]}$$

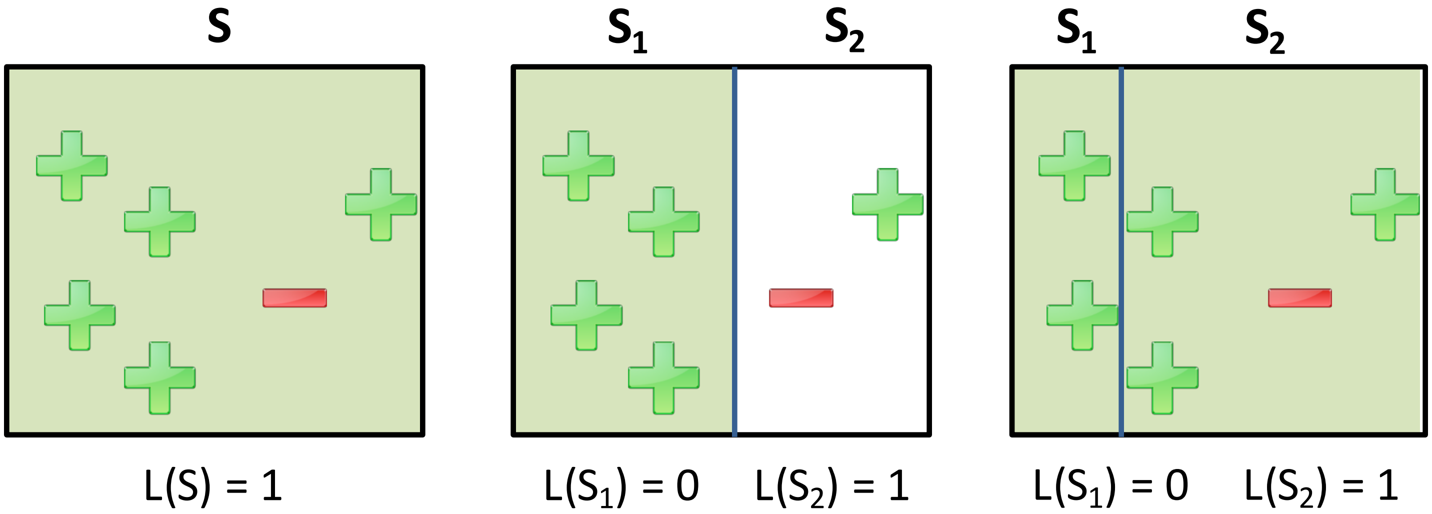
Classification Error on S'

Problems with 0/1 Loss

- What split best reduces impurity?

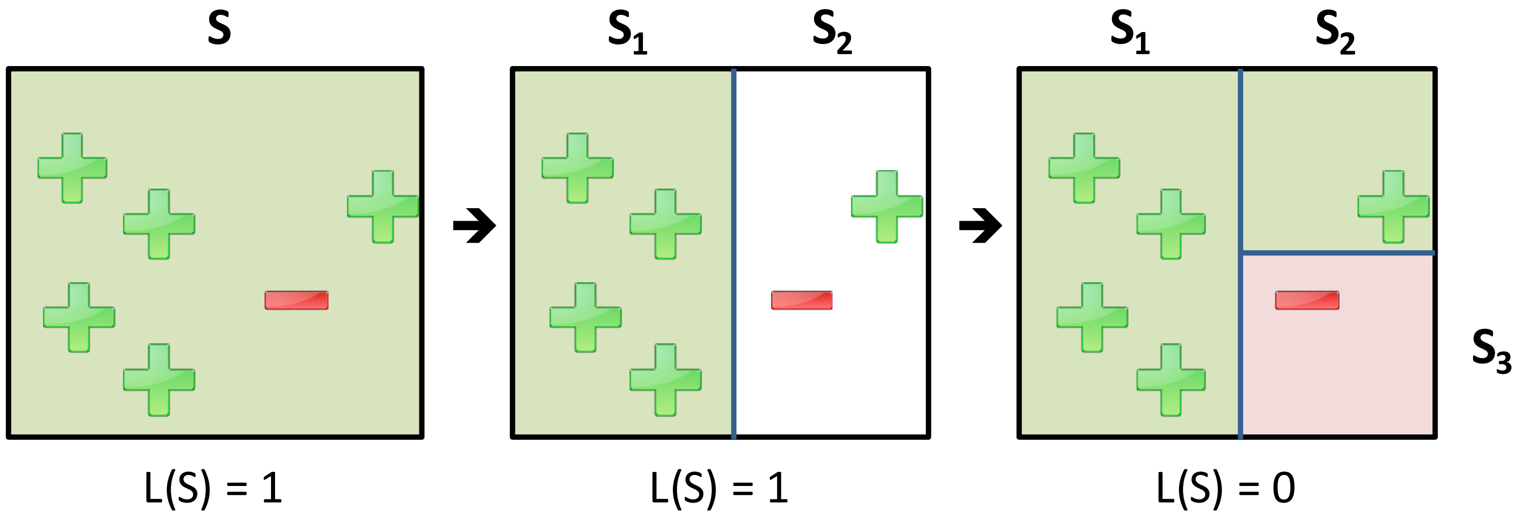
$$L(S') = \min_{\hat{y}\in\{0,1\}} \sum_{(x,y)\in S'} 1_{[\hat{y}\neq y]}$$

All Partitionings Give Same Impurity Reduction!



Problems with 0/1 Loss

- 0/1 Loss is discontinuous
- A good partitioning may not improve 0/1 Loss…
  - E.g., leads to an accurate model with subsequent split…

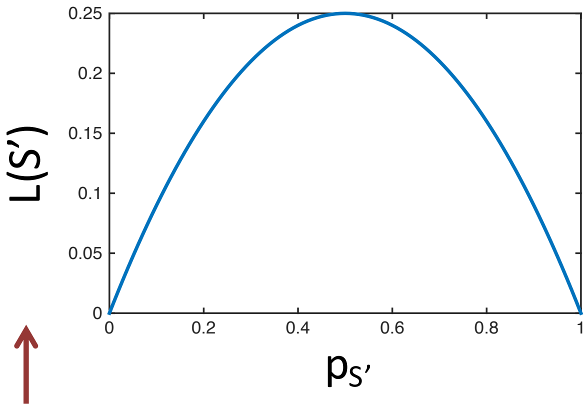


Surrogate Impurity Measures

- Want more continuous impurity measure
- First try: Bernoulli Variance:

$$L(S') = |S'|p_{S'}(1-p_{S'}) = \frac{\#pos * \#neg}{|S'|}$$

$p_{S'}$ = fraction of S' that are positive examples



Assuming |S'|=1

Worst Purity

$P = 1/2$, $L(S') = |S'|*1/4$

$P = 1$, $L(S') = |S'|*0$

$P = 0$, $L(S') = |S'|*0$

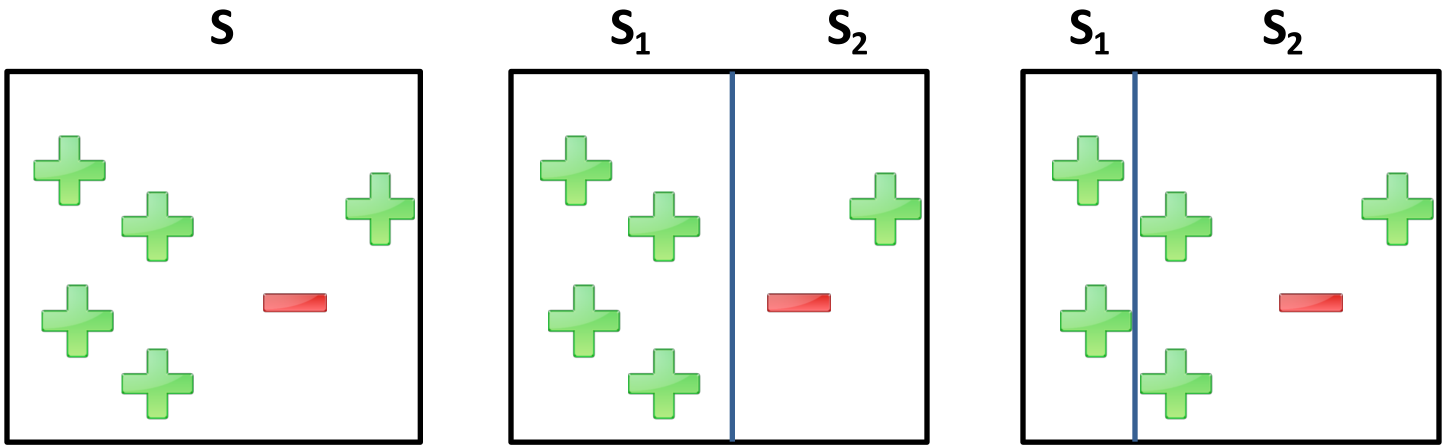
Perfect Purity

Bernoulli Variance as Impurity

- What split best reduces impurity?

$$L(S') = \left|S'\right| p_{S'}(1 - p_{S'}) = \frac{\# pos * \# neg}{|S'|}$$

$p_{S'}$ = fraction of S' that are positive examples



L(S) = 5/6

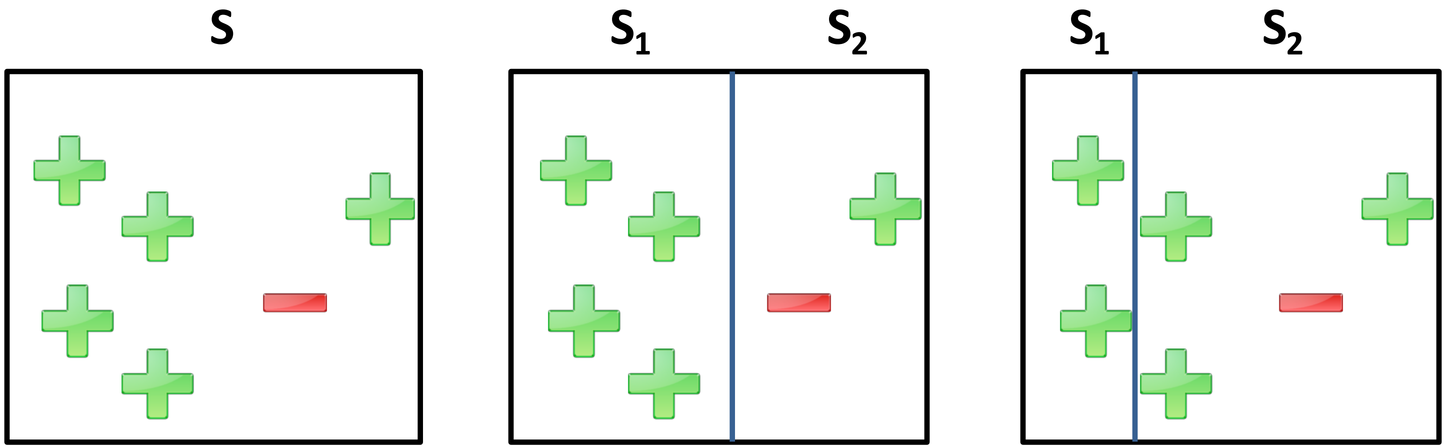
$L(S_1) = 0$ $L(S_2) = 1/2$

Best!

$L(S_1) = 0$ $L(S_2) = 3/4$

Interpretation of Bernoulli Variance

- Each partition = distribution over y
  - y is Bernoulli distributed with expected value $p_{S'}$
  - **Goal:** partitioning where each y has low variance



L(S) = 5/6 $L(S_1) = 0$ $L(S_2) = 1/2$ $L(S_1) = 0$ $L(S_2) = 3/4$

Best!

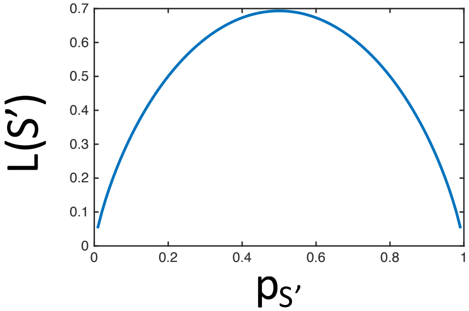
Other Impurity Measures

Define: 0*log(0) = 0

- Entropy: $L(S') = -|S'|\left(p_{S'}\log p_{S'} + \left(1-p_{S'}\right)\log\left(1-p_{S'}\right)\right)$
  - **aka: Information Gain:**

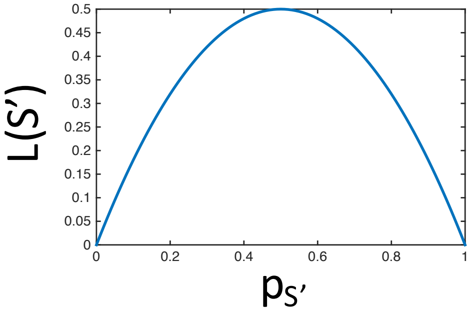
$$IG(A,B\,|\,S') = L(S') - L(A) - L(B)$$

  - (aka: Entropy Impurity Reduction)
  - Most popular.



- Gini Index:

$$L(S') = |S'|\left(1 - p_{S'}^2 - \left(1-p_{S'}\right)^2\right)$$



See also: http://www.ise.bgu.ac.il/faculty/liorr/hbchap9.pdf
(Terminology is slightly different.)

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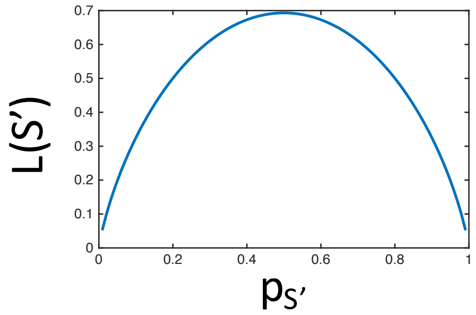
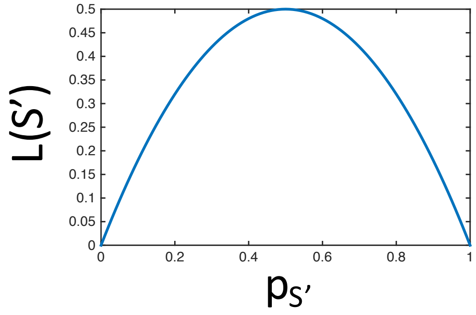


Most Good Impurity Measures Look Qualitatively The Same!



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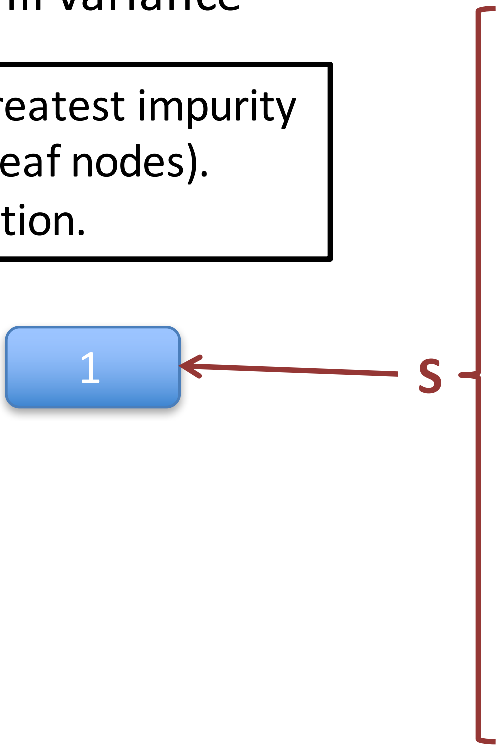
Top-Down Training

- Define impurity measure L(S')
  - E.g., L(S') = Bernoulli Variance

Loop: Choose split with greatest impurity reduction (over all leaf nodes).
Repeat: until stopping condition.

Step 1:
L(S) = 12/7



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Gena	10	0	0

x y

See TreeGrowing (Fig 9.2) in http://www.ise.bgu.ac.il/faculty/liorr/hbchap9.pdf

Top-Down Training

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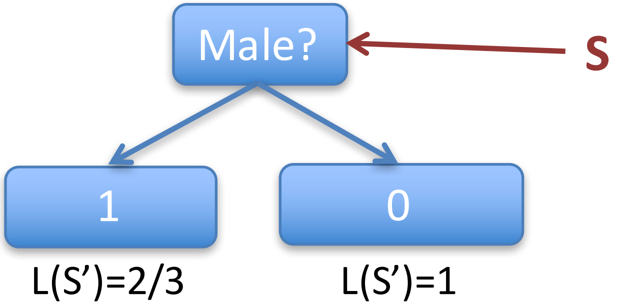
Step 1:
L(S) = 12/7

Step 2:
L(S) = 5/3



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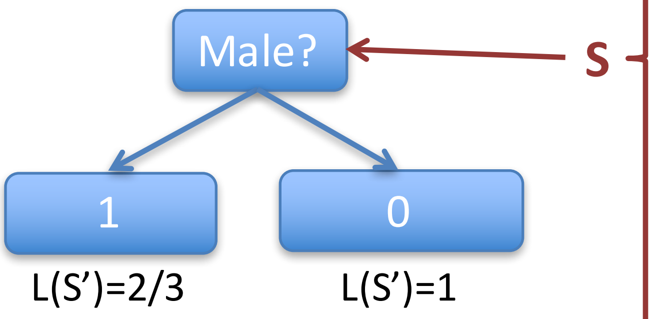

Step 3: Loop over all leaves, find best split.

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x y

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Repeat: until stopping condition.

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Step 2:
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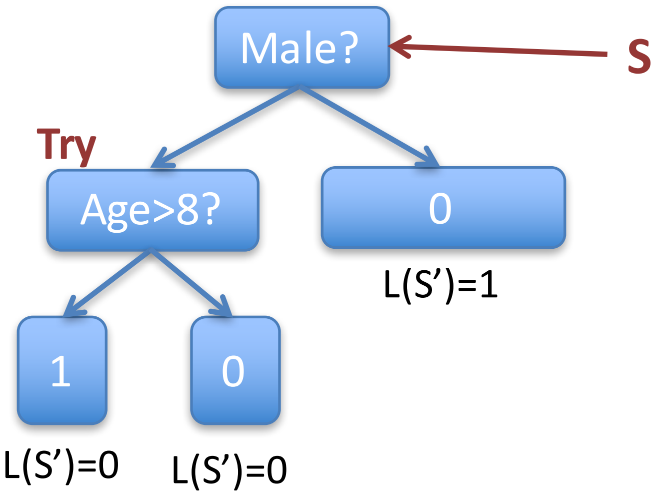
Step 3:
L(S) = 1



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x

y

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Top-Down Training

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Step 1:
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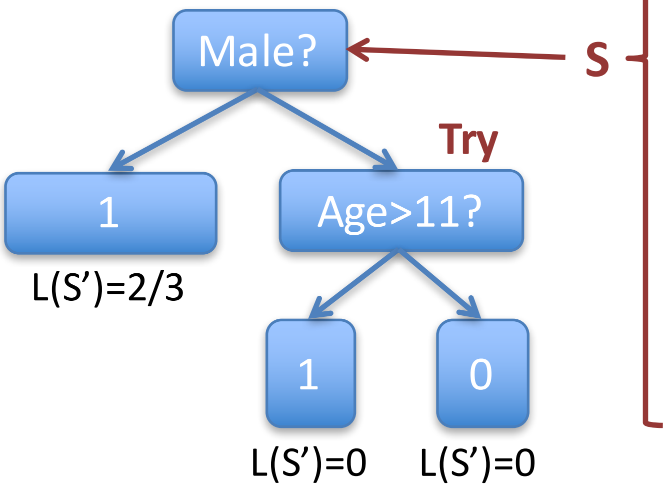
Step 2:
L(S) = 5/3

Step 3:
L(S) = 2/3



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Step 2:
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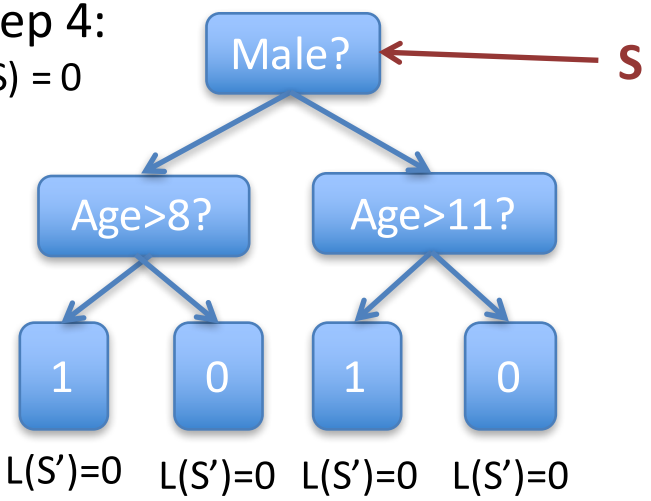
Step 3:
L(S) = 2/3

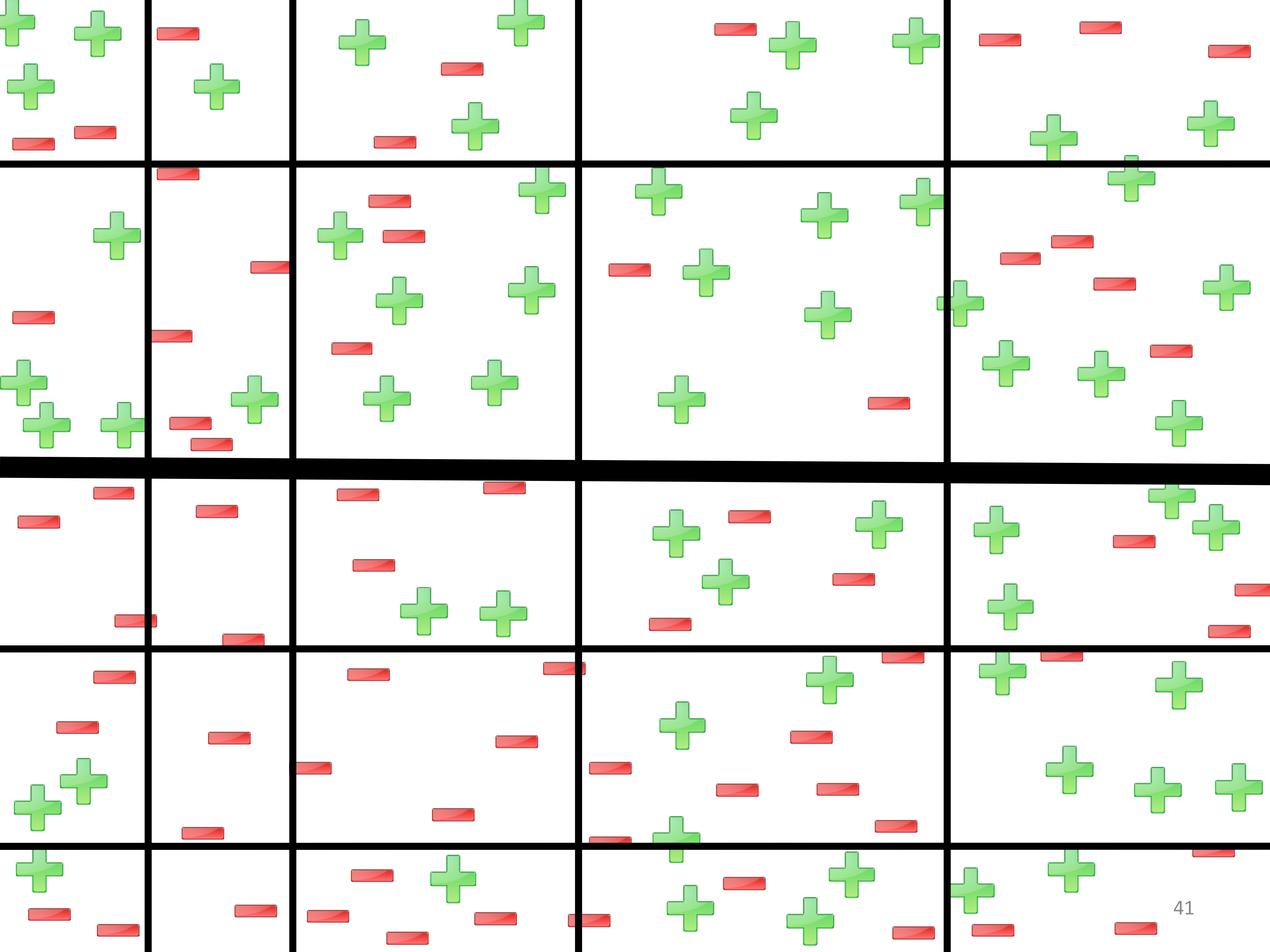
Step 4:
L(S) = 0

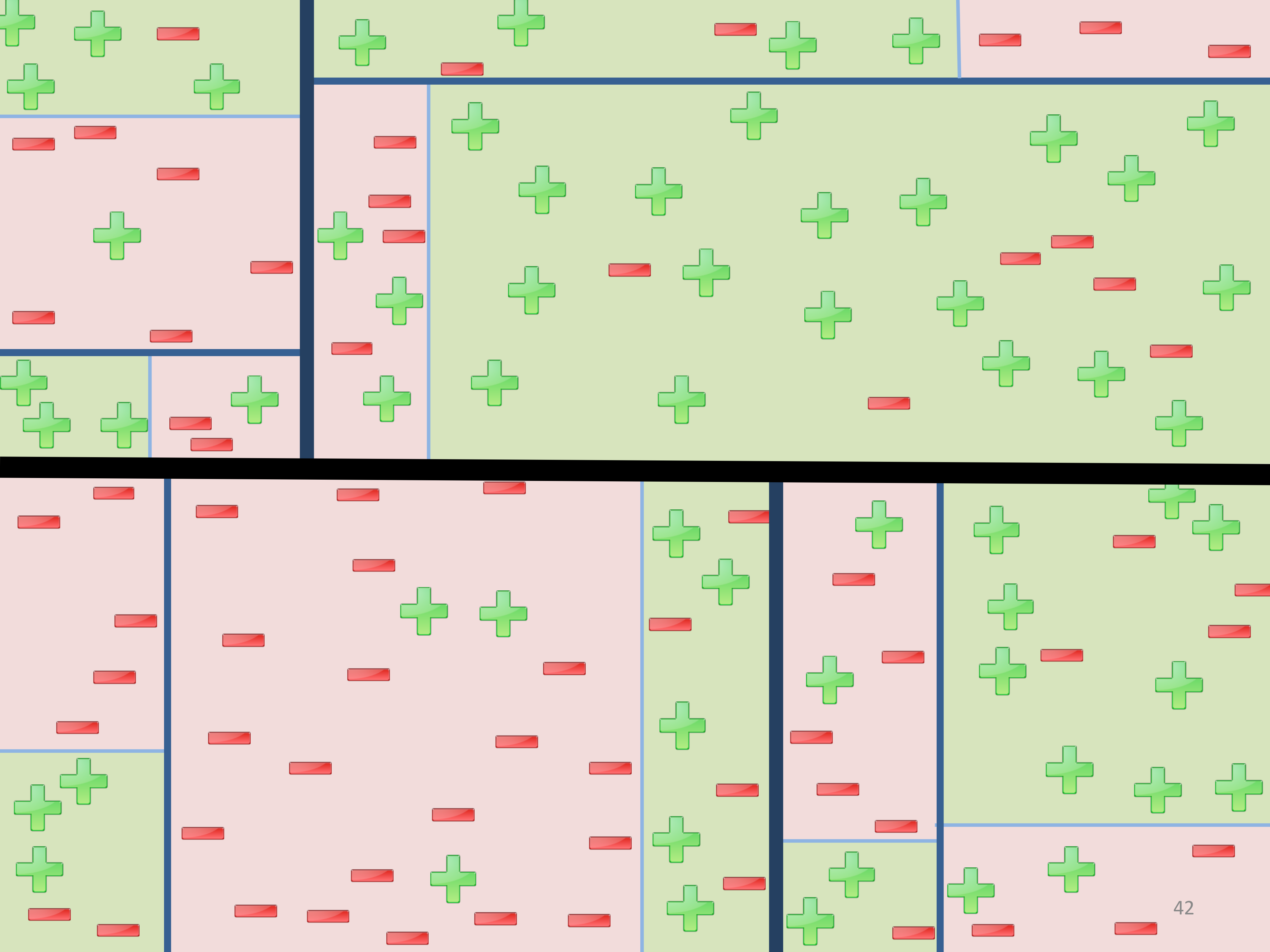


Name	Age	Male?	Height > 55"
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	10	0	0

x y

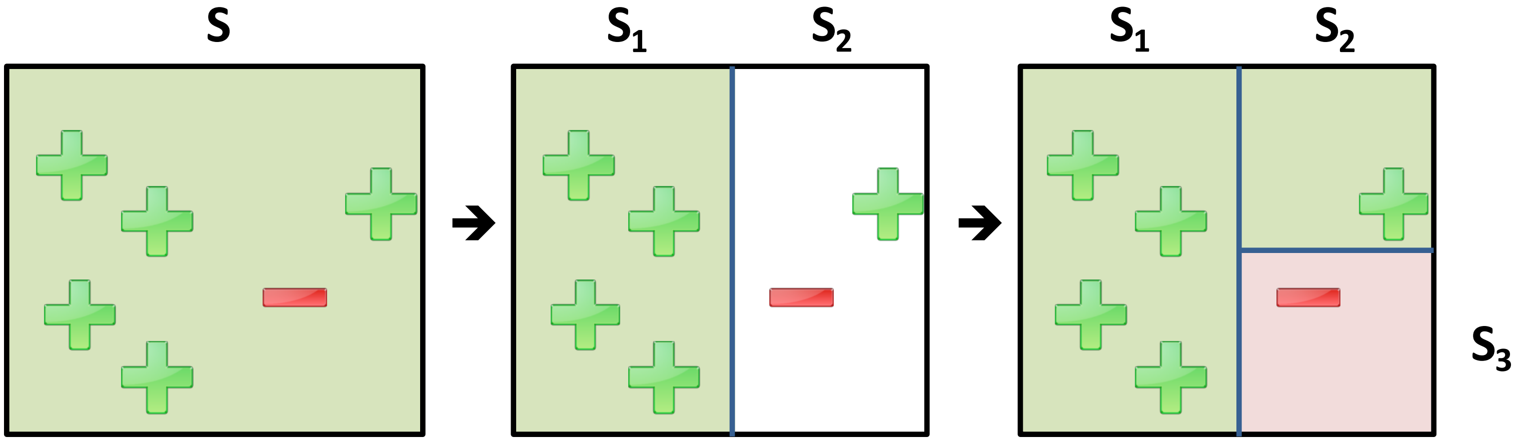
See TreeGrowing (Fig 9.2) in http://www.ise.bgu.ac.il/faculty/liorr/hbchap9.pdf





Properties of Top-Down Training

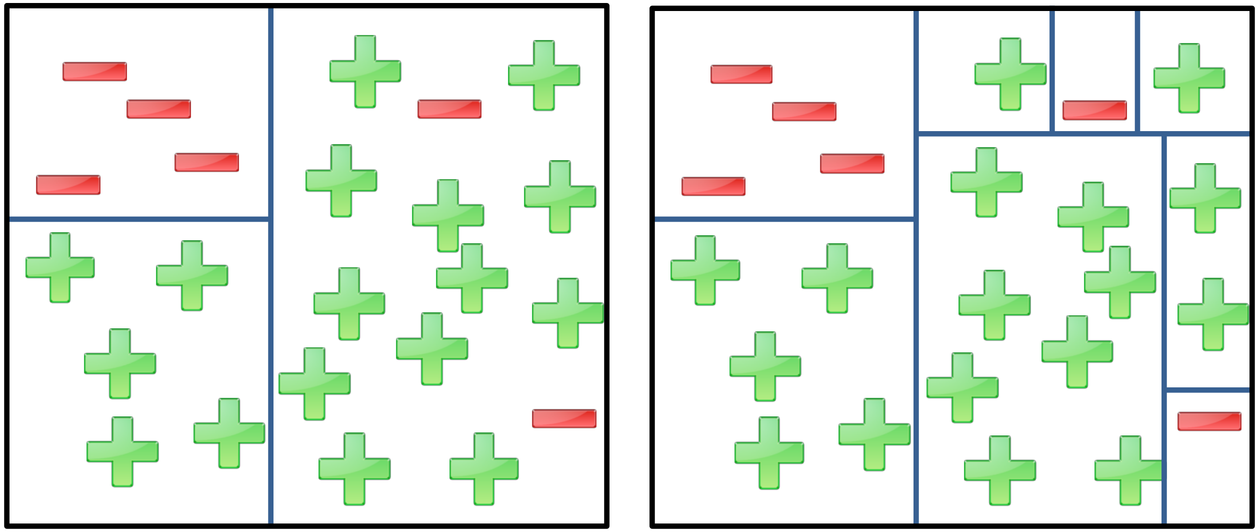
- Every intermediate step is a decision tree
  - You can stop any time and have a model
- Greedy algorithm
  - Doesn't backtrack
  - Cannot reconsider different higher-level splits.



When to Stop?

- If kept going, can learn tree with zero training error.
  - But such tree is probably overfitting to training set.
- How to stop training tree earlier?
  - I.e., how to regularize?

Which one has better test error?

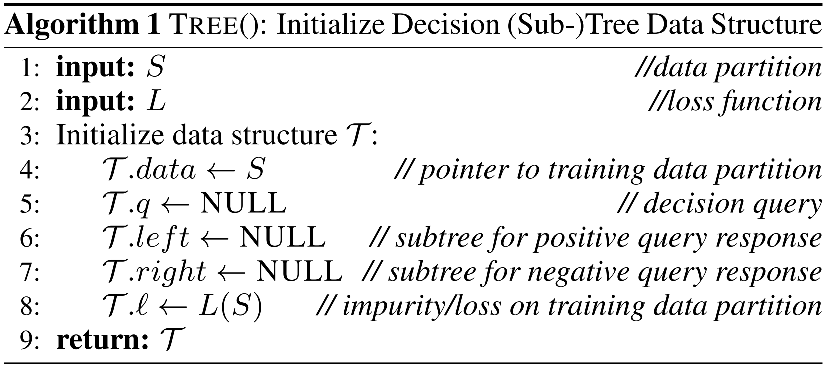


Stopping Conditions (Regularizers)

- **Minimum Size:** do not split if resulting children are smaller than a minimum size.
- **Maximum Depth:** do not split if the resulting children are beyond some maximum depth of tree.
- **Maximum #Nodes:** do not split if tree already has maximum number of allowable nodes.
- **Minimum Reduction in Impurity:** do not split if resulting children do not reduce impurity by at least δ%.

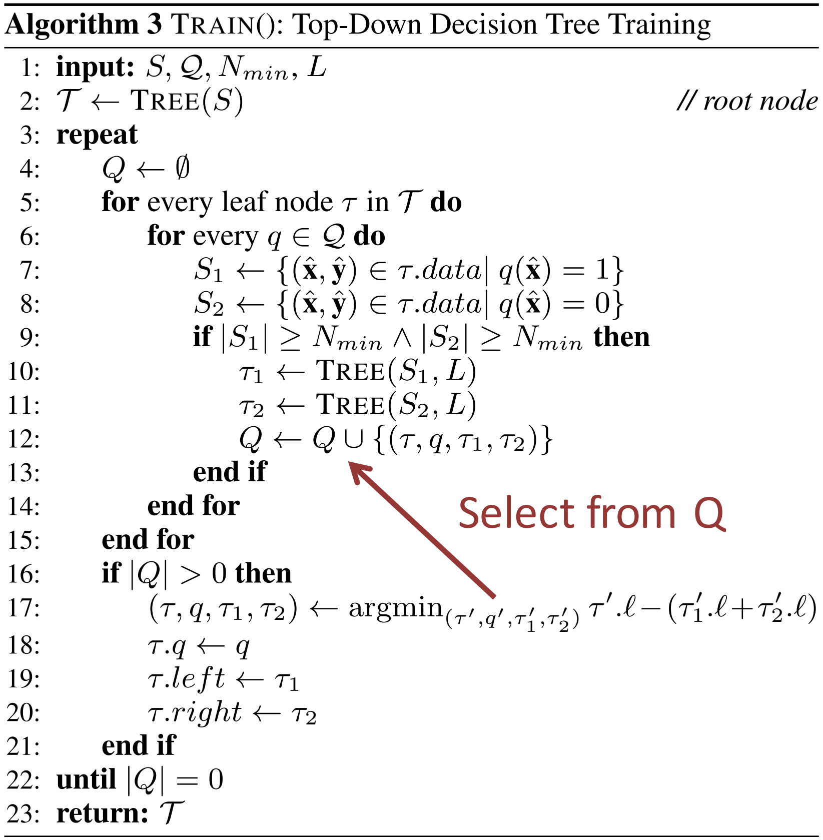
See also, Section 5 in: http://www.ise.bgu.ac.il/faculty/liorr/hbchap9.pdf

Pseudocode for Training



Algorithm 1 TREE(): Initialize Decision (Sub-)Tree Data Structure

```
input: $S$                                   //data partition
input: $L$                                   //loss function
Initialize data structure $\mathcal{T}$:
    $\mathcal{T}.data \leftarrow S$          // pointer to training data partition
    $\mathcal{T}.q \leftarrow$ NULL          // decision query
    $\mathcal{T}.left \leftarrow$ NULL       // subtree for positive query response
    $\mathcal{T}.right \leftarrow$ NULL      // subtree for negative query response
    $\mathcal{T}.\ell \leftarrow L(S)$       // impurity/loss on training data partition
return: $\mathcal{T}$
```



Algorithm 3 TRAIN(): Top-Down Decision Tree Training

```
input: $S, \mathcal{Q}, N_{min}, L$
$\mathcal{T} \leftarrow$ TREE($S$)                              // root node
repeat
    $Q \leftarrow \emptyset$
    for every leaf node $\tau$ in $\mathcal{T}$ do
        for every $q \in \mathcal{Q}$ do
            $S_1 \leftarrow \{(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \tau.data |\ q(\hat{\mathbf{x}}) = 1\}$
            $S_2 \leftarrow \{(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \tau.data |\ q(\hat{\mathbf{x}}) = 0\}$
            if $|S_1| \geq N_{min} \wedge |S_2| \geq N_{min}$ then
                $\tau_1 \leftarrow$ TREE($S_1, L$)
                $\tau_2 \leftarrow$ TREE($S_2, L$)
                $Q \leftarrow Q \cup \{(\tau, q, \tau_1, \tau_2)\}$
            end if
        end for
    end for
    if $|Q| > 0$ then
        $(\tau, q, \tau_1, \tau_2) \leftarrow \text{argmin}_{(\tau', q', \tau_1', \tau_2')}\ \tau'.\ell - (\tau_1'.\ell + \tau_2'.\ell)$
        $\tau.q \leftarrow q$
        $\tau.left \leftarrow \tau_1$
        $\tau.right \leftarrow \tau_2$
    end if
until $|Q| = 0$
return: $\mathcal{T}$
```

Select from Q

Stopping condition is minimum leaf node size: N_{min}

See TreeGrowing (Fig 9.2) in http://www.ise.bgu.ac.il/faculty/liorr/hbchap9.pdf

Classification vs Regression

Classification	Regression
Labels are {0,1}	Labels are Real Valued
Predict Majority Class in Leaf Node	Predict Mean of Labels in Leaf Node
Piecewise Constant Function Class	Piecewise Constant Function Class
Goal: Minimize 0/1 Loss	Goal: Minimize squared loss
Impurity Based on Fraction of Positives vs Negatives	Impurity = Squared Loss

Recap: Decision Tree Training

- Train Top-Down
  - Iteratively split existing leaf node into 2 leaf nodes
- Minimize Impurity (= Training Loss)
  - E.g., Entropy
- Until Stopping Condition (= Regularization)
  - E.g., Minimum Node Size
- Finding optimal tree is intractable
  - E.g., tree satisfying minimal leaf sizes with lowest impurity.

Recap: Decision Trees

- Piecewise Constant Model Class
  - Non-linear!
  - Axis-aligned partitions of feature space
- Train to minimize impurity of training data in leaf partitions
  - Top-Down Greedy Training
- Often more accurate than linear models
  - If enough training data

Bagging

(Bootstrap Aggregation)

Outline

- Recap: Bias/Variance Tradeoff
- Bagging
  - Method for minimizing variance
  - Not specific to Decision Trees
- Random Forests
  - Extension of Bagging
  - Specific to Decision Trees

Outline

- **Recap: Bias/Variance Tradeoff**

- Bagging
  - Method for minimizing variance
  - Not specific to Decision Trees

- Random Forests
  - Extension of Bagging
  - Specific to Decision Trees

Test Error

- **“True” distribution:** P(x,y)
  - Unknown to us
- **Train:** $h_S(x) = y$
  - Using training data: $S = \{(x_i, y_i)\}_{i=1}^{N}$
  - Sampled from P(x,y)
- **Test Error:**

$$L_P(h_S) = E_{(x,y)\sim P(x,y)}\left[L(y, h_S(x))\right]$$

- **Overfitting:** Test Error >> Training Error

True Distribution P(x,y)

Person	Age	Male?	Height > 55”
James	11	1	1
Jessica	14	0	1
Alice	14	0	1
Amy	12	0	1
Bob	10	1	1
Xavier	9	1	0
Cathy	9	0	1
Carol	13	0	1
Eugene	13	1	0
Rafael	12	1	1
Dave	8	1	0
Peter	9	1	0
Henry	13	1	0
Erin	11	0	0
Rose	7	0	0
Iain	8	1	1
Paulo	12	1	0
Margaret	10	0	1
Frank	9	1	1
Jill	13	0	0
Leon	10	1	0
Sarah	12	0	0
Gena	8	0	0
Patrick	5	1	1
	⋮		

Training Set S

Person	Age	Male?	Height > 55”	
Alice	14	0	1	✓
Bob	10	1	1	✓
Carol	13	0	1	✓
Dave	8	1	0	✓
Erin	11	0	0	✗
Frank	9	1	1	✗
Gena	8	0	0	✓
			y	h(x)

Test Error:

$$\mathcal{L}(h) = E_{(x,y)\sim P(x,y)}[\ L(h(x),y)\ ]$$

Bias-Variance Decomposition

$$E_S\left[L_P(h_S)\right] = E_S\left[E_{(x,y)\sim P(x,y)}\left[L(y,h_S(x))\right]\right]$$

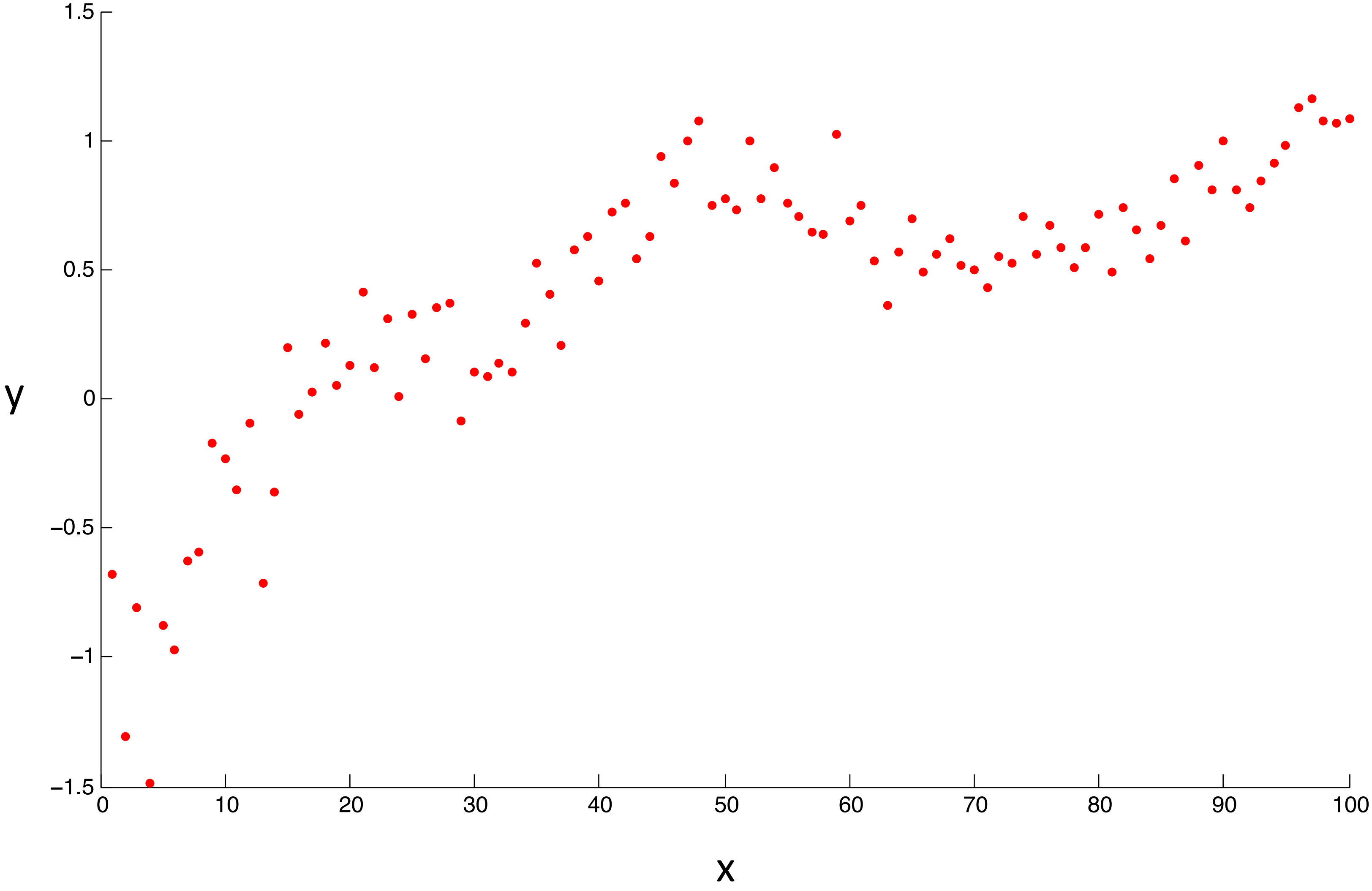
- For squared error:

$$E_S\left[L_P(h_S)\right] = E_{(x,y)\sim P(x,y)}\left[\underbrace{E_S\left[\left(h_S(x)-H(x)\right)^2\right]}_{\text{Variance Term}} + \underbrace{\left(H(x)-y\right)^2}_{\text{Bias Term}}\right]$$

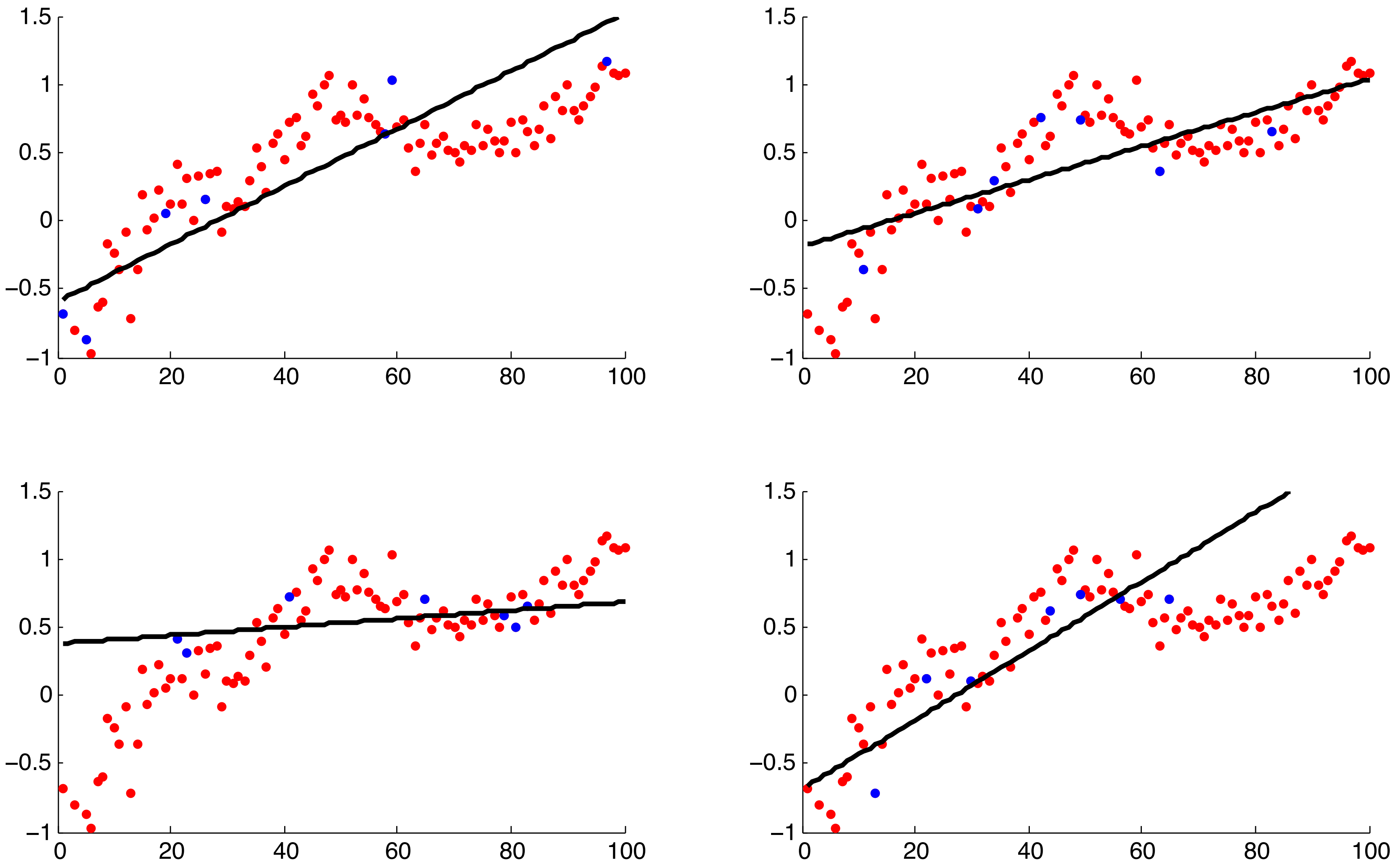
$$H(x) = \underset{\uparrow}{E_S}\left[h_S(x)\right]$$

“Average prediction on x”

Example P(x,y)

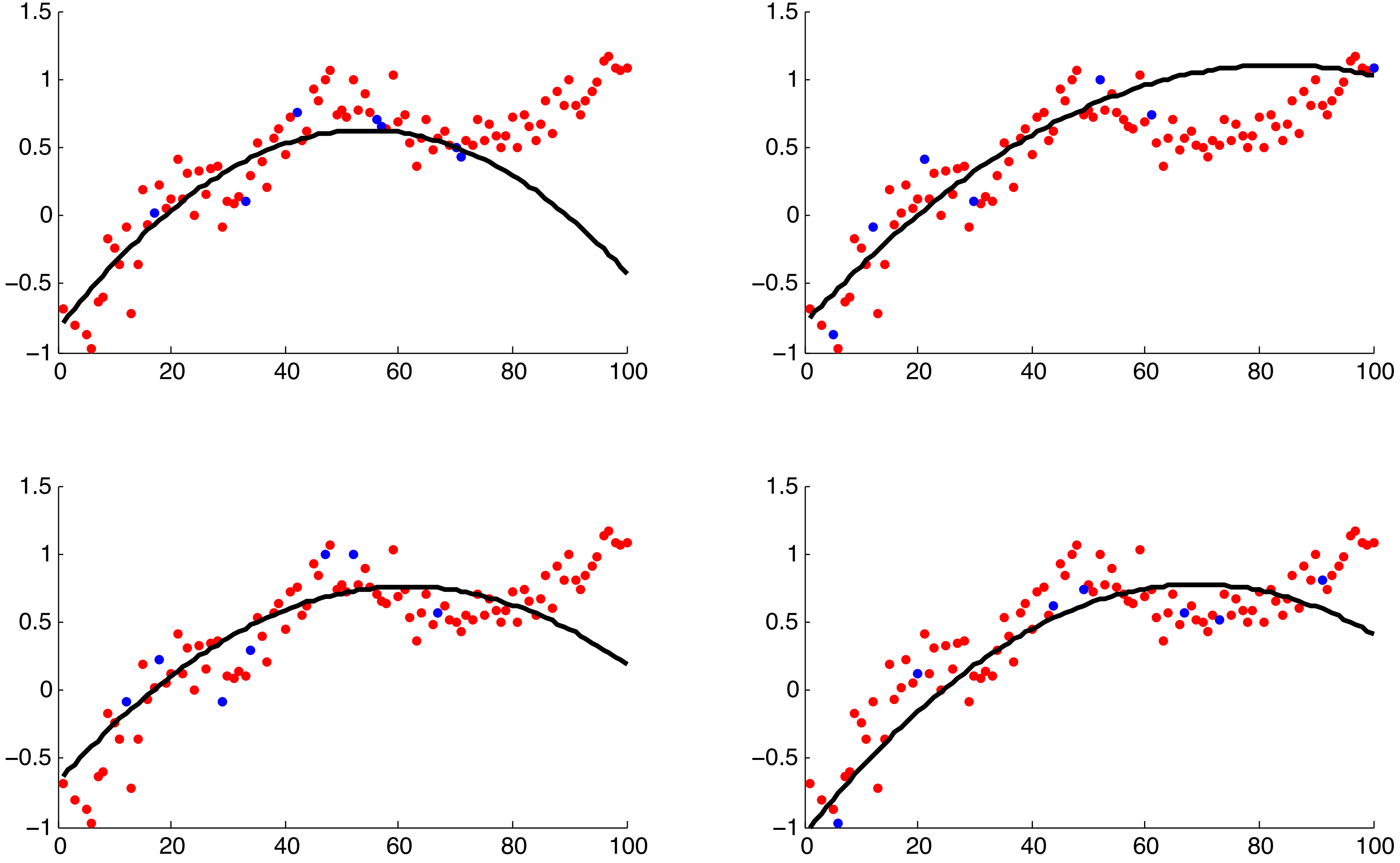


$h_S(x)$ Linear

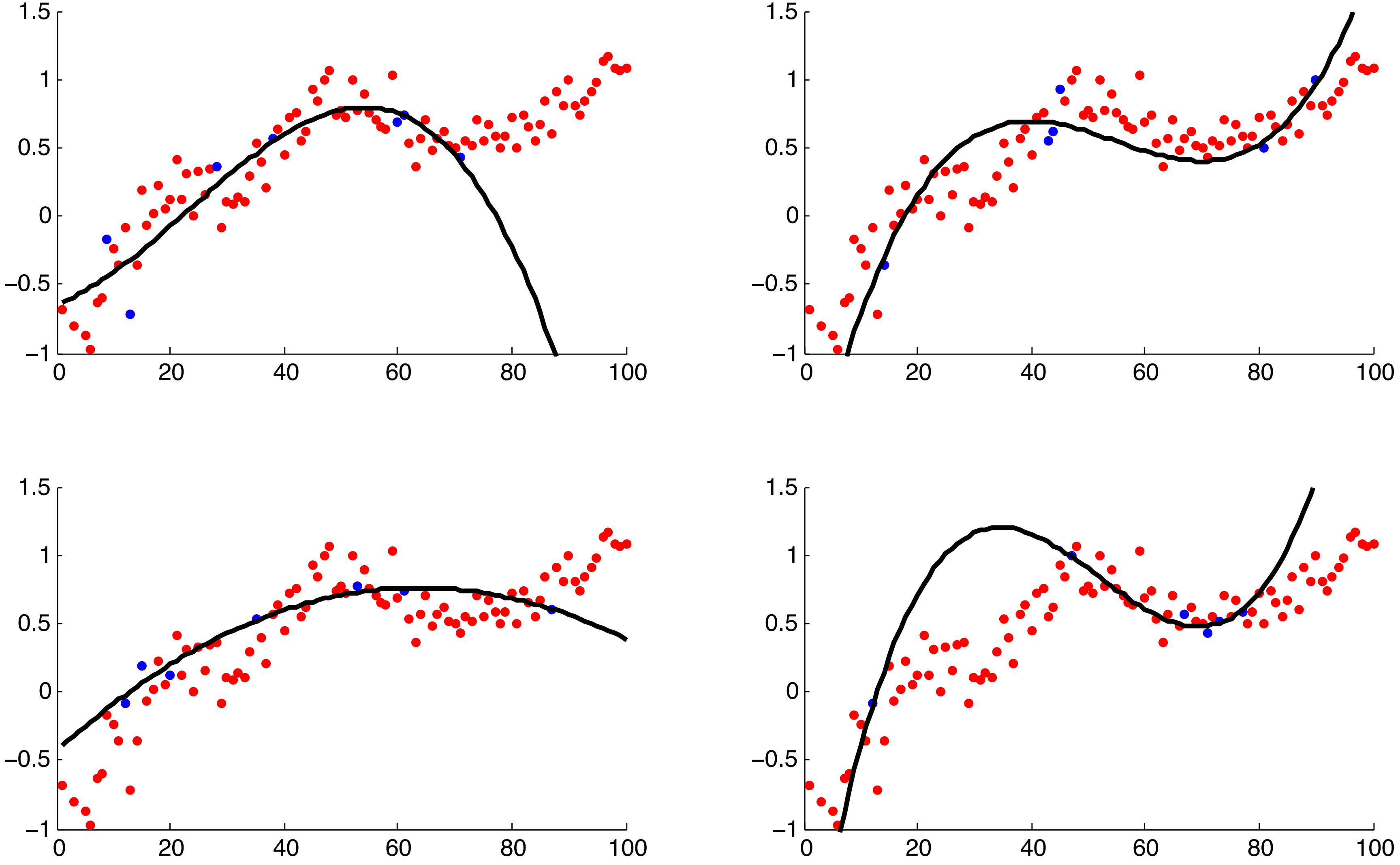


1.5
1
0.5
0
−0.5
−1
0
20
40
60
80
100
1.5
1
0.5
0
−0.5
−1
0
20
40
60
80
100
1.5
1
0.5
0
−0.5
−1
0
20
40
60
80
100
1.5
1
0.5
0
−0.5
−1
0
20
40
60
80
100

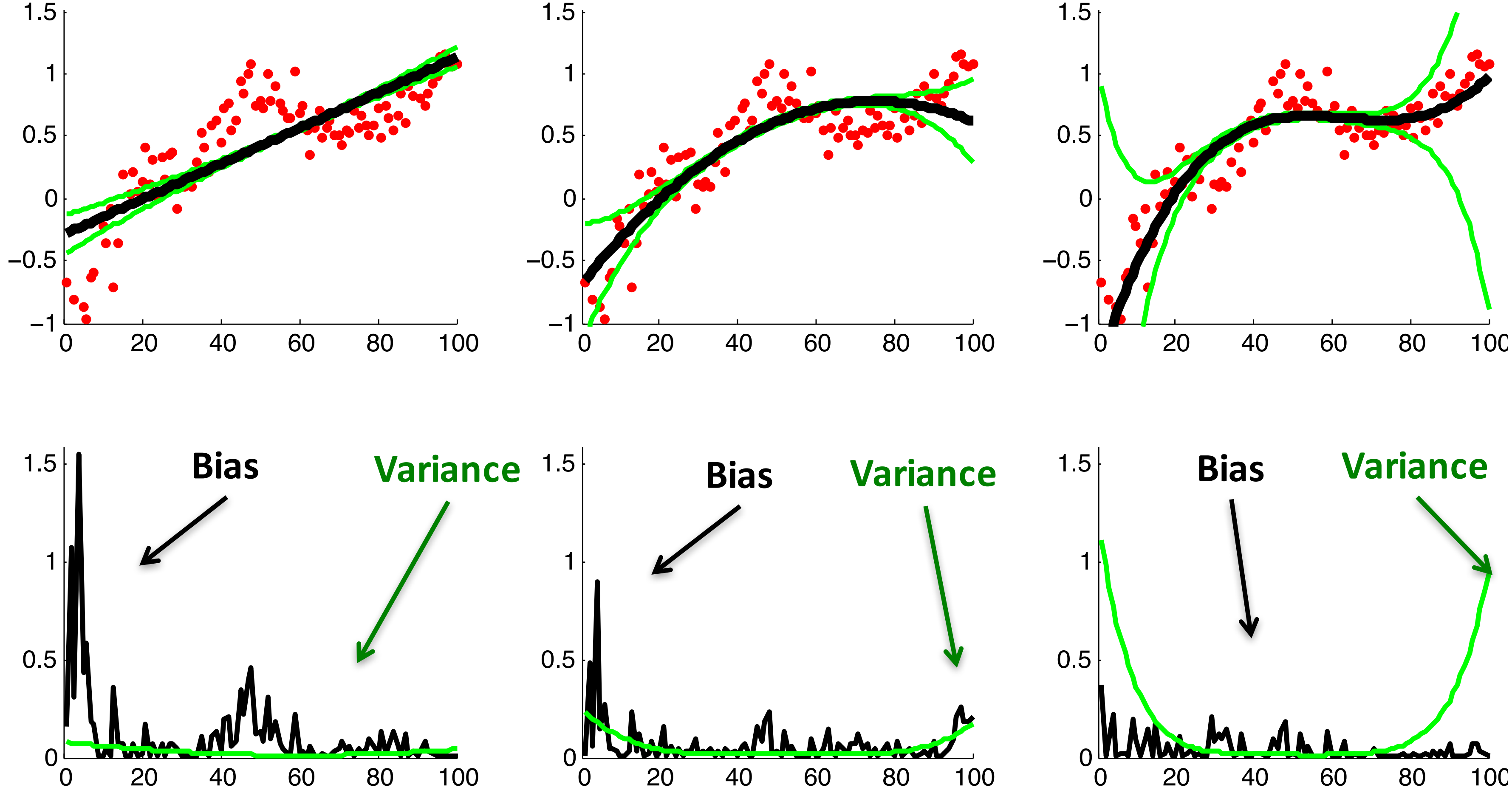
$h_S(x)$ Quadratic



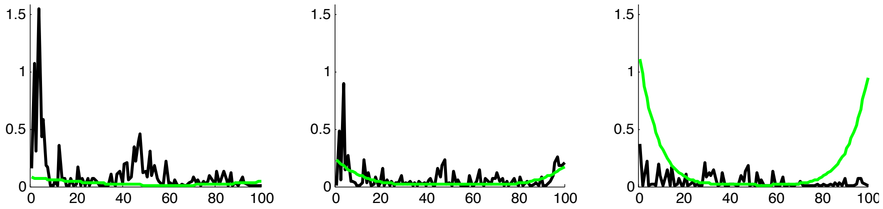
$h_S(x)$ Cubic



Bias-Variance Trade-off

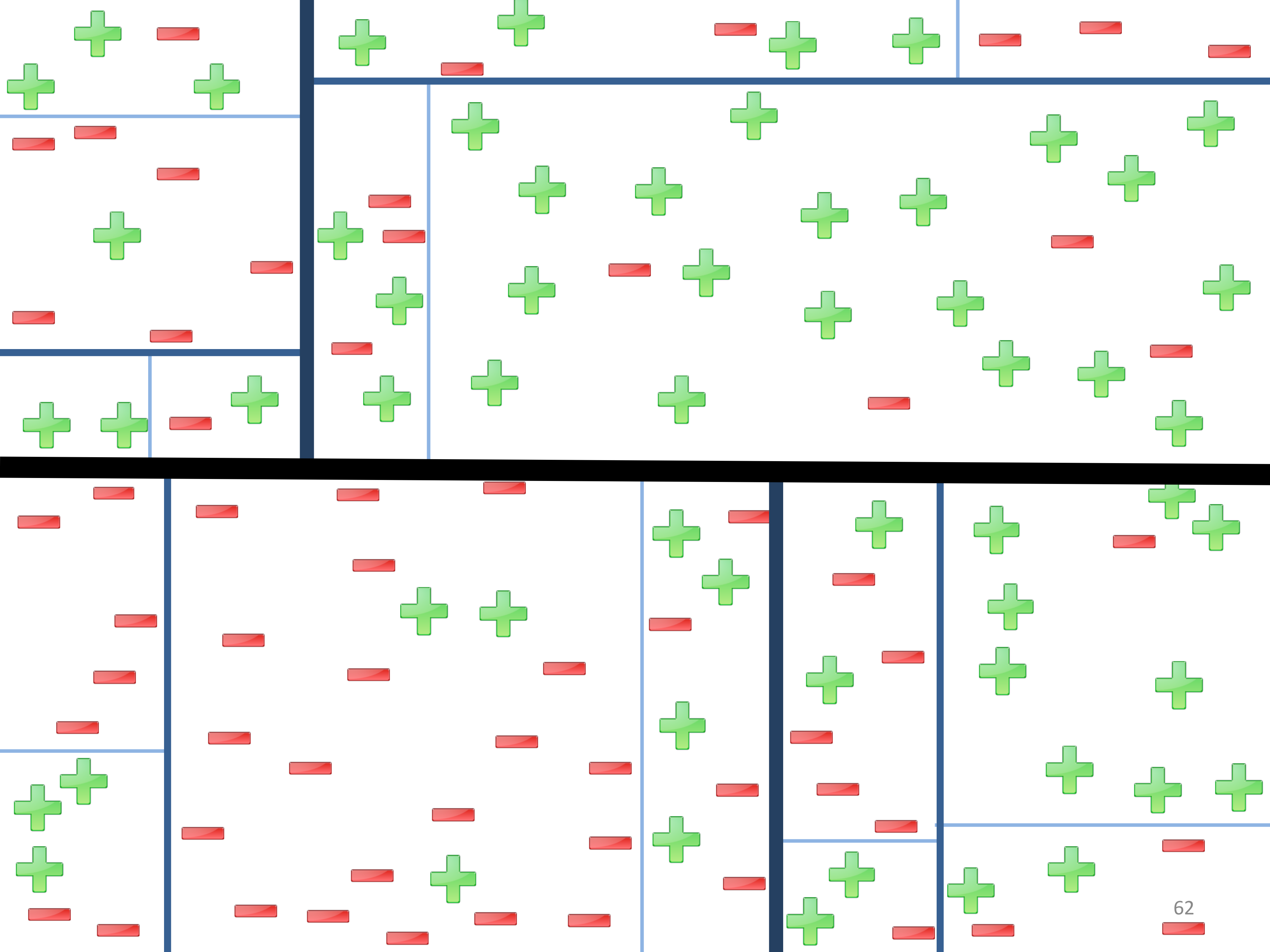


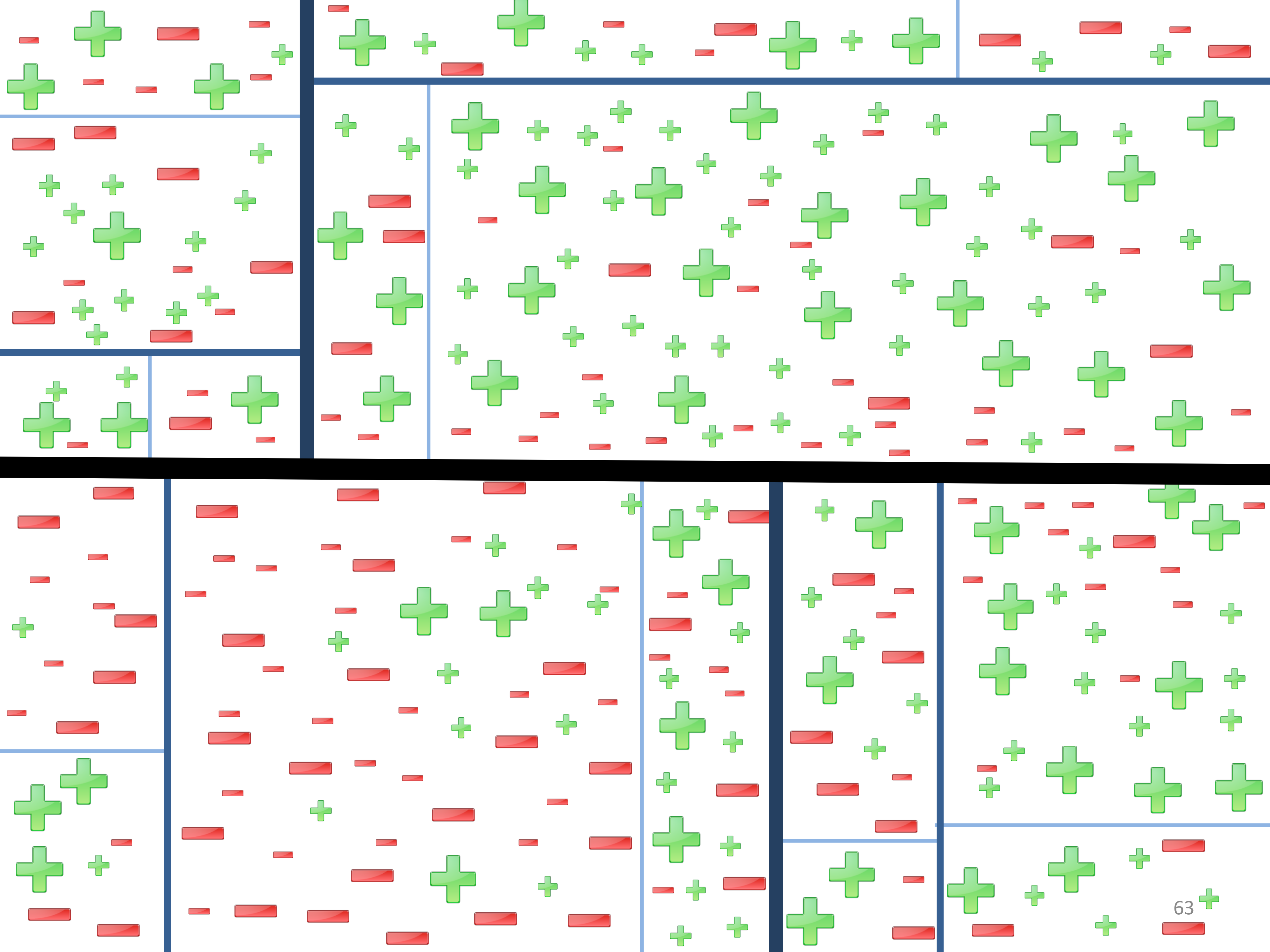
Overfitting vs Underfitting



- High variance implies **overfitting**
  - Model class unstable
  - Variance increases with model complexity
  - Variance reduces with more training data.

- High bias implies **underfitting**
  - Even with no variance, model class has high error
  - Bias decreases with model complexity
  - Independent of training data size





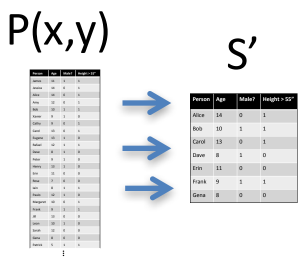
Decision Trees are Low Bias, High Variance Models

Unless you Regularize a lot…
…but then often worse than Linear Models

Highly Non-Linear, Can Easily Overfit

Different Training Samples Can Lead to Very Different Trees

Bagging



Person	Age	Male?	Height > 55”
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	8	0	0

sampled independently

- **Goal:** reduce variance
- **Ideal setting:** many training sets S’
  - Train model using each S’
  - Average predictions

Variance reduces linearly
Bias unchanged

$$\boxed{E_S[(h_S(x) - y)^2] = E_S[(Z-\check{z})^2] + \check{z}^2}$$

Expected Error
On single (x,y)

Variance

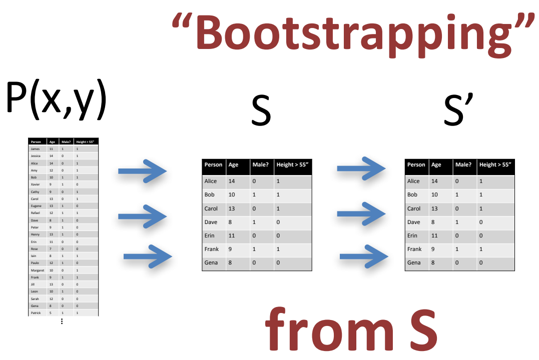
Bias

$Z = h_S(x) – y$

$\check{z} = E_S[Z]$

“Bagging Predictors” [Leo Breiman, 1994]
http://statistics.berkeley.edu/sites/default/files/tech-reports/421.pdf

Bagging



P(x,y)

Person	Age	Male?	Height > 55"
James	11	1	1
Jessica	14	0	1
Alice	14	0	1
Amy	12	0	1
Bob	10	1	1
Xavier	9	1	0
Cathy	9	0	1
Carol	13	0	1
Eugene	13	1	0
Rafael	12	1	1
Dave	8	1	0
Peter	9	1	0
Henry	13	1	0
Erin	11	0	0
Rose	7	0	0
Iain	8	1	1
Paulo	12	1	0
Margaret	10	0	1
Frank	9	1	1
Jill	13	0	0
Leon	10	1	0
Sarah	12	0	0
Gena	8	0	0
Patrick	5	1	1

⋮

S

Person	Age	Male?	Height > 55"
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	8	0	0

S'

Person	Age	Male?	Height > 55"
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	8	0	0

- **Goal:** reduce variance

- **In practice:** resample S' with replacement
  - Train model using each S'
  - Average predictions

Variance reduces sub-linearly
(Because S' are correlated)
Bias often increases slightly

$$\boxed{\underbrace{E_S[(h_S(x) - y)^2]}_{\substack{\text{Expected Error} \\ \text{On single (x,y)}}} = \underset{\substack{\uparrow \\ \textbf{Variance}}}{E_S[(Z-\check{z})^2]} + \underset{\substack{\uparrow \\ \textbf{Bias}}}{\check{z}^2}}$$

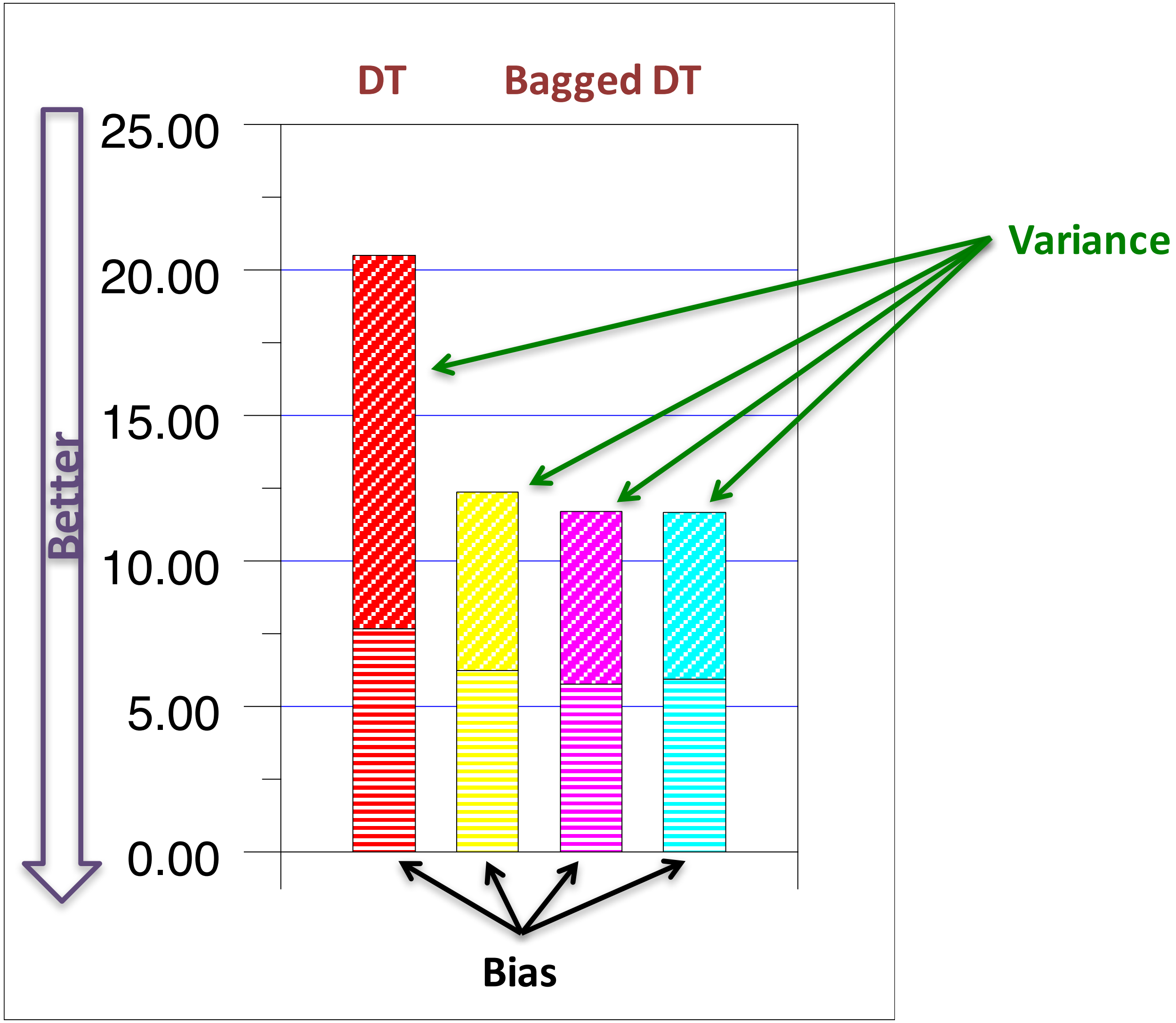
$$Z = h_S(x) – y$$
$$\check{z} = E_S[Z]$$

Bagging = Bootstrap Aggregation

"Bagging Predictors" [Leo Breiman, 1994]
http://statistics.berkeley.edu/sites/default/files/tech-reports/421.pdf

Recap: Bagging for DTs

- **Given:** Training Set S
- **Bagging:** Generate Many Bootstrap Samples S'
  - Sampled with replacement from S
    - |S'| = |S|
  - Train Minimally Regularized DT on S'
    - High Variance, Low Bias
- **Final Predictor:** Average of all DTs
  - Averaging reduces variance



“An Empirical Comparison of Voting Classification Algorithms: Bagging, Boosting, and Variants”
Eric Bauer & Ron Kohavi, Machine Learning 36, 105–139 (1999)
http://ai.stanford.edu/~ronnyk/vote.pdf

Why Bagging Works

- Define Ideal Aggregation Predictor $h_A(x)$:
  - Each S' drawn from true distribution P

$$h_A(x) = E_{S \sim P(x,y)}\left[\underset{\text{Decision Tree Trained on S}}{h_S(x)}\right]$$

- We will first compare the error of $h_A(x)$ vs $h_S(x)$
- Then show how to adapt comparison to Bagging

"Bagging Predictors" [Leo Breiman, 1994]
http://statistics.berkeley.edu/sites/default/files/tech-reports/421.pdf

Analysis of Ideal Aggregate Predictor

(Squared Loss)

$$h_A(x) = E_{S\sim P(x,y)}\left[h_S(x)\right]$$

Decision Tree Trained on S

$$\underbrace{E_S\left[L\left(y,h_S(x)\right)\right]}_{\text{Expected Loss of } h_S \text{ on single } (x,y)} = E_S\left[\left(y-h_S(x)\right)^2\right]$$

Linearity of Expectation

$$= E_S\left[y^2\right] - 2E_S\left[yh_S(x)\right] + E_S\left[h_S(x)^2\right]$$

$$= y^2 - 2yE_S\left[h_S(x)\right] + \boxed{E_S\left[h_S(x)^2\right]}$$

$E[Z^2] \geq E[Z]^2$

$(Z=h_{S'}(x))$

$$\geq y^2 - 2yE_S\left[h_S(x)\right] + \boxed{E_S\left[h_S(x)\right]^2}$$

Definition of h_A

$$= y^2 - 2yh_A(x) + h_A(x)^2$$

$$= \left(y-h_A(x)\right)^2$$

$$= L\left(y,h_A(x)\right)$$

Loss of h_A

“Bagging Predictors” [Leo Breiman, 1994]

http://statistics.berkeley.edu/sites/default/files/tech-reports/421.pdf

Key Insight

- Ideal Aggregate Predictor Improves if:

$$E_S\left[h_S(x)^2\right] > E_S\left[h_S(x)\right]^2 = h_A(x)^2$$

Large improvement if $h_S(x)$ is “unstable” (high variance)

$h_A(x)$ is guranteed to be at least as good as $h_S(x)$.

- Bagging Predictor Improves if:

$$E_S\left[h_S(x)^2\right] > E_S\left[E_{S'\sim S}\left[h_{S'}(x)\right]^2\right] = E_S\left[h_B(x)^2\right]$$

Improves if $h_B(x)$ is much more stable than $h_S(x)$

$h_B(x)$ can sometimes be more unstable than $h_S(x)$

Bias of $h_B(x)$ can be worse than $h_S(x)$.

“Bagging Predictors” [Leo Breiman, 1994]
http://statistics.berkeley.edu/sites/default/files/tech-reports/421.pdf

Random Forests

Random Forests

- **Goal:** reduce variance
  - Bagging can only do so much
  - Resampling training data asymptotes

- **Random Forests:** sample data & features!
  - Sample S'
  - Train DT
    - At each node, sample features
  - Average predictions

Further de-correlates trees

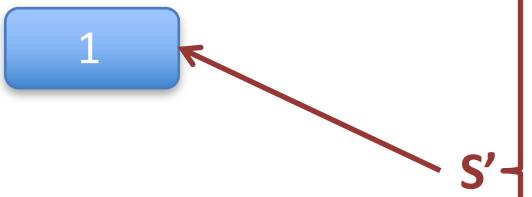
"Random Forests – Random Features" [Leo Breiman, 1997]
http://oz.berkeley.edu/~breiman/random-forests.pdf

Top-Down Random Forest Training

Loop: Sample T random splits at each Leaf. Choose split with greatest impurity reduction.

Repeat: until stopping condition.

Step 1:



Name	Age	Male?	Height > 55”
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Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	10	0	0

x y

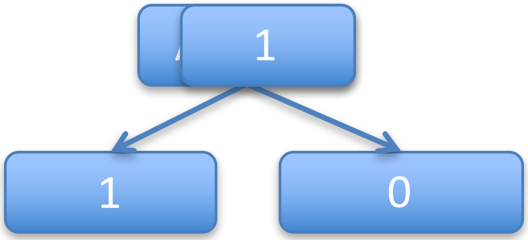
“Random Forests – Random Features” [Leo Breiman, 1997]
http://oz.berkeley.edu/~breiman/random-forests.pdf

Top-Down Random Forest Training

Loop: Sample T random splits at each Leaf. Choose split with greatest impurity reduction.
Repeat: until stopping condition.

Step 1:

Step 2:



Randomly decide only look at age,
Not gender.

Name	Age	Male?	Height > 55”
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	10	0	0

S’

x

y

“Random Forests – Random Features” [Leo Breiman, 1997]
http://oz.berkeley.edu/~breiman/random-forests.pdf

Top-Down Random Forest Training

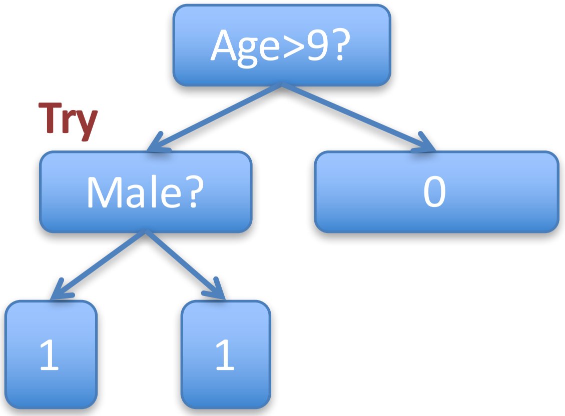
Loop: Sample T random splits at each Leaf. Choose split with greatest impurity reduction.

Repeat: until stopping condition.

Step 1:

Step 2:

Step 3:



Randomly decide only look at gender.

S’

Name	Age	Male?	Height > 55”
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	10	0	0

x y

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Top-Down Random Forest Training

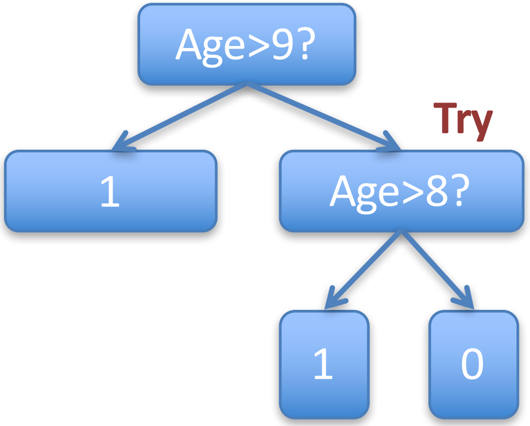
Loop: Sample T random splits at each Leaf. Choose split with greatest impurity reduction.

Repeat: until stopping condition.

Step 1:

Step 2:

Step 3:



Randomly decide only look at age.

Name	Age	Male?	Height > 55”
Alice	14	0	1
Bob	10	1	1
Carol	13	0	1
Dave	8	1	0
Erin	11	0	0
Frank	9	1	1
Gena	10	0	0

S’

x

y

“Random Forests – Random Features” [Leo Breiman, 1997]
http://oz.berkeley.edu/~breiman/random-forests.pdf

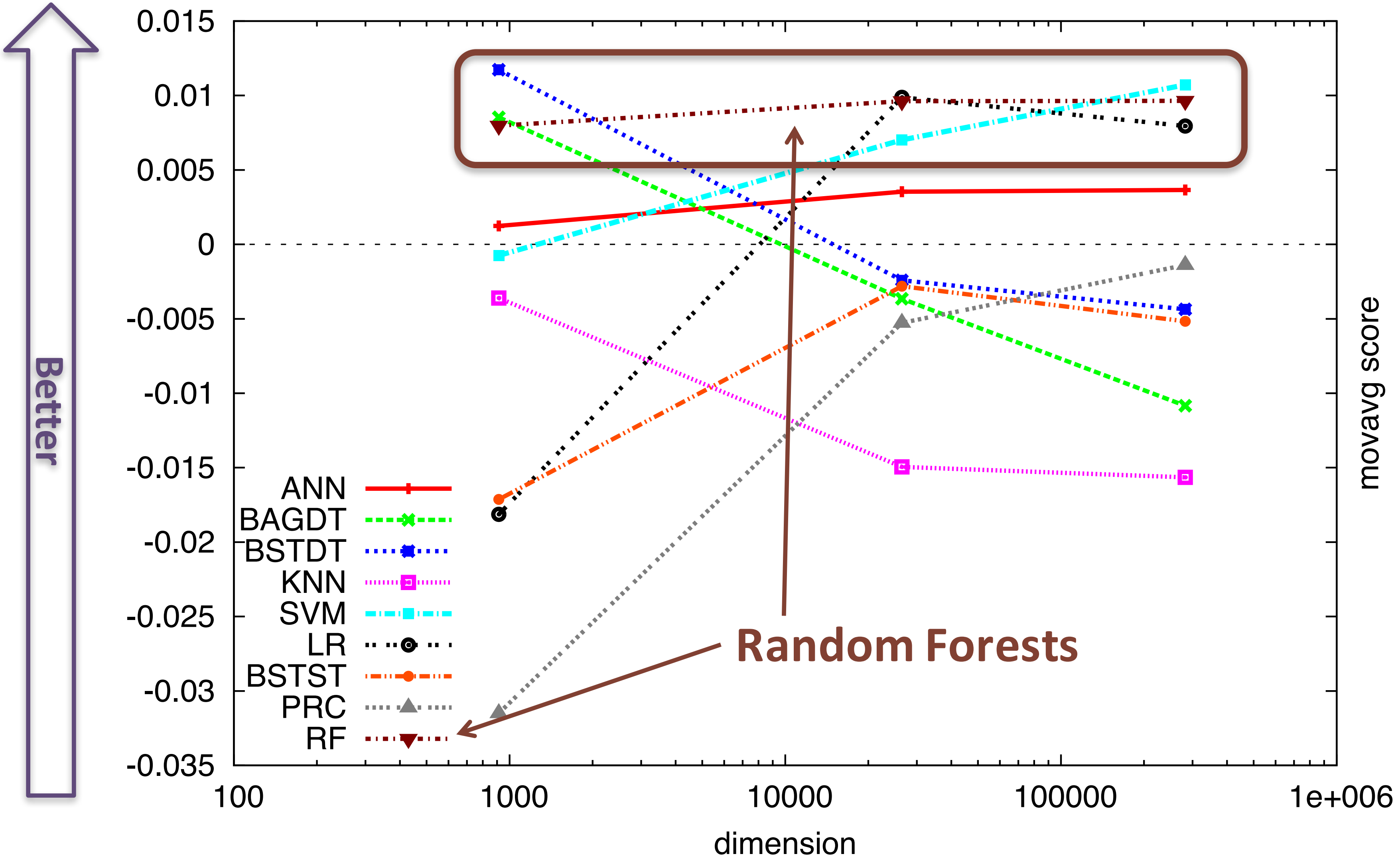
Recap: Random Forests

- Extension of Bagging to sampling Features

- Generate Bootstrap S’ from S
  - – Train DT Top-Down on S’
  - – Each node, sample subset of features for splitting
    - Can also sample a subset of splits as well

- Average Predictions of all DTs

“Random Forests – Random Features” [Leo Breiman, 1997]
http://oz.berkeley.edu/~breiman/random-forests.pdf



Average performance over many datasets
Random Forests perform the best

"An Empirical Evaluation of Supervised Learning in High Dimensions"
Caruana, Karampatziakis & Yessenalina, ICML 2008

Next Lecture

- Boosting
  - Method for reducing bias
- Ensemble Selection
  - Very general method for combining classifiers
  - Multiple-time winner of ML competitions
- Recitation Next Week:
  - Deep Learning Tutorial (Keras)